Problem Solving (MA2201)

Week 12

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1. Show that each rational number x is uniquely expressible as a finite sum of the form
$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$
where $a_1, a_2, \ldots, a_n$ are integers with $0 \leq a_r < r$ for $r = 2, 3, \ldots, n$.

2. Each of four people A, B, C, D tell the truth just 1 time in 3. A makes a statement, and B says that C says that D says that A is telling the truth. What is the probability that A is actually telling the truth?

3. Suppose α, β are positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$. Show that the sequences
$$[\alpha], [2\alpha], [3\alpha], \ldots \quad \text{and} \quad [\beta], [2\beta], [3\beta], \ldots$$
together contain each positive integer just once.

4. Find all pairs of distinctive positive rationals x, y such that
$$x^y = y^x.$$

5. Show that if a, b, c are positive real numbers then
$$[(a+b)(b+c)(c+a)]^{1/3} \geq \frac{2}{\sqrt{3}}(ab+bc+ca)^{1/2}.$$

6. Bob and Alice arrange to meet between 1pm and 2pm. Each agrees to wait just 15 minutes for the other. What is the probability that they meet?

7. If

$$N = \overbrace{111\ldots1}^{1000\ 1\text{'s}},$$

what is the 1000th digit after the decimal point of $\sqrt{N}$?

8. Show that a continuous function $f : [0,1] \to [0,1]$ must leave at least one point fixed: $f(x) = x$.

9. Determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

10. Determine $\det A$, where A is the $n \times n$ matrix with entries

$$a_{ij} = \frac{1}{x_i + y_j}.$$

11. What is the maximal area of a quadrilateral with sides $1, 2, 3, 4$?

12. Can you find an equilateral triangle all of whose vertices have integer coordinates?

13. Given any two polynomials $f(t), g(t)$, show that there exists a non-zero polynomial $F(x, y)$ such that $F(f(t), g(t)) = 0$ identically.

14. Show that the equation

$$y^5 = x^2 + 4$$

has no integer solutions.

15. Show that there exists a real number α such that the fractional part of α^n lies between $1/3$ and $2/3$ for all positive integers n.

Challenge Problem

Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < \left|h\sqrt{m} - k\sqrt{n}\right| < 2\epsilon.$$