Problem Solving (MA2201)

Week 12

Timothy Murphy

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1. Show that each rational number x is uniquely expressible as a finite sum of the form

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

where $a_1, a_2, \ldots, a_n$ are integers with $0 \le a_r < r$ for $r = 2, 3, \ldots, n$.

Answer: *We denote the fractional part of the real number x by $\{x\}$, so that*

$$x = [x] + \{x\}.$$

Let us define $x_1, x_2, \ldots; a_1, a_2, \ldots; s_1, s_2, \ldots$ successively as follows:

$$\begin{aligned}
&a_1 = [x], x_1 = \{x\}, s_1 = a_1;\\
&a_2 = [2x_1], x_2 = \{2x_1\}, s_2 = s_1 + a_2/2!;\\
&\ldots\\
&a_r = [rx_{r-1}], x_r = \{rx_{r-1}\}, s_r = s_{r-1} + a_r/r!\\
&\ldots.
\end{aligned}$$

Then

$$rx_{r-1} = a_r + x_r,$$

and

$$0 \le x_{r-1} < 1 \implies 0 \le a_r < r.$$

Also, it follows by induction that

$$x = s_r + \frac{x_r}{r!}.$$

This is true when $r = 1$; and if it is true for $r - 1$ then

$$\begin{aligned} x &= s_{r-1} + \frac{x_{r-1}}{(r-1)!} \\ &= s_{r-1} + \frac{r x_{r-1}}{r!} \\ &= s_{r-1} + \frac{a_r + x_r}{r!} \\ &= s_r + \frac{x_r}{r!}. \end{aligned}$$

To see that the series terminates if x is rational, evidently

$$n!x \in \mathbb{Z}$$

for some n. Let n be the least positive integer with this property.

We can see by induction that

$$r!s_r \in \mathbb{Z}.$$

For this is true when $r = 1$; and if it is true for $r - 1$ then

$$r!s_r = r!s_{r-1} + a_r \in \mathbb{Z}.$$

Now

$$n!x = n!s_n + \frac{x_n}{.}$$

Since $n!x, n!s_n \in \mathbb{Z}$ it follows that $x_n \in \mathbb{Z}$. But $0 \le \{x\} < 1$. Hence

$$\{x_n\} = 0 \implies a_{n+1} = 0,$$

so the series terminates at this point.

Finally, to show that the series is unique, suppose

$$x = a_1 + \frac{a_2}{2!} + \cdots + \frac{a_n}{n!} = b_1 + \frac{b_2}{2!} + \cdots + \frac{b_m}{m!}.$$

We can assume that $a_n, b_m \neq 0$.

We argue by induction on $\max(m,n)$*. The result is trivially true if* $m = n = 1$*. Suppose* $m \neq n$*, say* $m < n$*. Multiplying the equation by* $(n-1)!$,

$$\frac{a_n}{n} \in \mathbb{Z},$$

and so $a_n = 0$*. Again, if* $m = n$ *then multiplying the equation by* $(n-1)!$,

$$\frac{a_n}{n} - \frac{b_n}{n} \in \mathbb{Z},$$

and so $a_n = b_n$ *Thus*

$$x = a_1 + \frac{a_2}{2!} + \cdots + \frac{a_{n-1}}{(n-1)!} = b_1 + \frac{b_2}{2!} + \cdots + \frac{b_{n-1}}{(n-1)!},$$

and it follows from the inductive hypothesis that $a_r = b_r$ *for all* r*.*

2. Each of four people A, B, C, D tell the truth just 1 time in 3. A makes a statement, and B says that C says that D says that A is telling the truth. What is the probability that A is actually telling the truth?

   Answer: *Let*

   $$a, b, c, d = \pm 1$$

   according as each of A, B, C, D *is telling the truth or not.*

   We are told that A *is telling the truth if*

   $$abcd = 1.$$

   Thus we have to compute the relative probability that $a = 1$ *given that* $abcd = 1$*. ie*

   $$\frac{\text{prob}(a = 1 \;\&\; bcd = 1)}{\text{prob}(abcd = 1)}.$$

   Now $abcd = 1$ *if either* $a = b = c = d = 1$ *or else just two of* a, b, c, d *is 1. We can choose these two in 6 ways. Thus*

   $$\begin{aligned}\text{prob}(abcd = 1) &= \left(\frac{1}{3}\right)^4 + 6\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 \\ &= \frac{25}{81}.\end{aligned}$$

Similarly $bcd = 1$ if either $b = c = d = 1$ or else just one of a, b, c is 1, which can be arranged in 3 ways. Thus

$$\begin{aligned}
\text{prob}(a = 1 \,\&\, bcd = 1) &= \text{prob}(a = 1) \cdot \text{prob}(bcd = 1) \\
&= \frac{1}{3}\left(\left(\frac{1}{3}\right)^3 + 3\frac{1}{3}\left(\frac{2}{3}\right)^2\right) \\
&= \frac{13}{81}.
\end{aligned}$$

Hence the relative probability is

$$\frac{13}{25}.$$

3. Suppose α, β are positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$. Show that the sequences

$$[\alpha], [2\alpha], [3\alpha], \ldots \quad \text{and} \quad [\beta], [2\beta], [3\beta], \ldots$$

together contain each positive integer just once.

Answer: *Clearly $\alpha, \beta > 1$. It follows that*

$$[(r + 1)\alpha] > [r\alpha],\ [(s + 1)\beta] > [s\beta].$$

So the integers in each of the two sequences are distinct.

We claim that no integer can occur in both the sequences. For suppose

$$[r\alpha] = [s\beta] = n.$$

Then

$$n < r\alpha < n + 1,\ n < s\beta < n + 1,$$

ie

$$\frac{n}{\alpha} < r < \frac{n + 1}{\alpha},\ \frac{n}{\beta} < s < \frac{n + 1}{\beta}.$$

(Note that we have inequality rather than equality on the left because α, β are irrational.) Adding,

$$n < r + s < n + 1,$$

which is impossible.

Next we show that there are $n-1$ elements of the two sequences in the range $[1, n-1]$ For suppose $[r\alpha], [s\beta$ are the largest elements of the two sequences in this range. Then

$$r\alpha < n < (r+1)\alpha,\ s\beta < n < (s+1)\beta,$$

ie

$$r < \frac{n}{\alpha} < r+1,\ s < \frac{n}{\beta} < s+1.$$

Adding,

$$r+s < n < r+s+2.$$

Hence

$$r+s = n-1.$$

Since the elements of the sequences in the range are all different, it follows that every number in the range appears in one sequence or the other; and so this is true for all positive integers.

4. Find all pairs of distinctive positive rationals x, y such that

$$x^y = y^x.$$

Answer: *Suppose*

$$x = \frac{a}{d}, y = \frac{b}{d} \text{ with } \gcd(a, b, d) = 1.$$

We may assume that $a > b$. (If $a = b$ then $x = y$.

Raising the equation to the dth power,

$$(a/d)^b = (b/d)^a.$$

Multiplying by d^{a+b},

$$a^b d^a = b^a d^b.$$

Hence

$$a^b d^{a-b} = b^a.$$

Evidently $b > 1$ *(since* $b = 1 \implies a = 1$*). So* $\gcd(a, d) = 1$ *(since otherwise* $\gcd(a, b, d) > 1$*). It follows that* b *splits into coprime factors* $b = b_1 b_2$ *with*

$$a^b = b_1^a,\ d^{a-b} = b_2^a.$$

Let

$$\gcd(a, b) = e,\ a = ea', b = eb', \gcd(a', b') = 1.$$

Note that if $\gcd(m, n) = 1$ *then*

$$a^m = b^n \implies a = c^n, b = c^m$$

for some $c \in \mathbb{N}$.

Thus

$$a = u^{a'}, b_1 = u^{b'}, d = v^{a'}, b_2 = v^{a'-b'}.$$

Now

$$e = \gcd(a, b) = u^{b'} = b_1 \implies a' = u^{a'-b'}, b' = b_2 = v^{a'-b'}.$$

Thus

$$a = u^{u^{a'-b'}}, b_1 = u^{v^{a'-b'}}$$

Let

$$a' - b' = n.$$

Then

$$a = u^{u^n}, b = u^{v^n} v^n, d = v^{u^n}, a' = u^n, b' = v^n.$$

Thus

$$n = u^n - v^n.$$

But if $n \geq 2$ *then*

$$n = u^n - v^n \geq (v+1)^n - v^n > nv^{n-1} \geq n.$$

Hence

$$n = 1;$$

and so

$$u - v = 1.$$

Thus

$a' = v+1, b' = v, e = (v+1)^v, a = (v+1)^{v+1}, b = (v+1)^v v, d = v^{v+1}.$

It seems that

$$x = \frac{a}{d} = \left(\frac{v+1}{v}\right)^{v+1}, \; y = \frac{b}{d} = \left(\frac{v+1}{v}\right)^{v},$$

does indeed give the general solution to the identity

$$x^y = y^x.$$

As an example, if we take $v = 2$ then

$$x = (3/2)^3, \; y = (3/2)^2,$$

and

$$x^{b'} = (3/2)^6 = y^{a'},$$

so that

$$x^b = y^a,$$

and

$$x^{b/d} = y^{a/d},$$

ie

$$x^y = y^x.$$

[As a particular case, we see that if x, y are integers then

$$d = 1 \implies v = 1, u = 2,$$

with just one solution

$$x = a = 2^2, \; y = b = 2;$$

that is,

$$4^2 = 2^4.$$

5. Show that if a, b, c are positive real numbers then

$$[(a+b)(b+c)(c+a)]^{1/3} \geq \frac{2}{\sqrt{3}}(ab+bc+ca)^{1/2}.$$

Answer:

6. Bob and Alice arrange to meet between 1pm and 2pm. Each agrees to wait just 15 minutes for the other. What is the probability that they meet?

Answer:

7. If

$$N = \overbrace{111\ldots1}^{1000\ 1\text{'s}},$$

what is the 1000th digit after the decimal point of $\sqrt{N}$?

Answer: *We have*

$$N = \frac{10^{1000}-1}{9}.$$

Thus

$$\begin{aligned}
\sqrt{N} &= \frac{\sqrt{10^{1000}-1}}{3} \\
&= \frac{10^{500}}{3}\left(1 - \frac{1}{10^{1000}}\right)^{1/2} \\
&= \frac{10^{500}}{3}\left(1 - \frac{1}{2}\frac{1}{10^{1000}} + \frac{3}{4}\frac{1}{10^{2000}} + \ldots\right).
\end{aligned}$$

So

$$\begin{aligned}
\sqrt{N} &= \frac{10^{500}}{3} - \frac{1}{6}\frac{1}{10^{500}} + \frac{1}{4}\frac{1}{10^{1500}} + \ldots \\
\sqrt{N} &= \frac{10^{500}}{3} - \frac{1}{6}\frac{1}{10^{500}} + \epsilon
\end{aligned}$$

where $0 < \epsilon < 1/10^{1500}$. *Thus*

$$\begin{aligned}
\sqrt{N} &= 33\ldots33.333\cdots - 0.\overbrace{00\ldots00}^{500\ 0\text{'s}}1666\cdots + \epsilon \\
&= 33\ldots33.\overbrace{33\ldots33}^{500\ 3\text{'s}}1666\cdots + \epsilon.
\end{aligned}$$

Thus the 1000th digit after the decimal point is 6.

8. Show that a continuous function $f : [0,1] \to [0,1]$ must leave at least one point fixed: $f(x) = x$.

Answer: *We have*

$$0 \le f(0),\ f(1) \le 1.$$

Let

$$S = \{x \in [0,1] : t \le f(t) \text{ for } 0 \le t \le x\}$$

Then S is not empty, since $0 \in S$. Let X be the upper bound of S. Then

$$f(X) = X.$$

For by the continuity of $f(x)$, if $f(X) > X$ then

$$f(x) > X > x$$

for $xin(X - \epsilon, X]$, contradicting the definition of X; and similarly if $f(X) < X$ then

$$f(x) < X < x$$

for $xin[X, X+\epsilon)$. again contradicting the definition of X.

9. Determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

Answer: *This question was a bit of a cheat, as it requires some simple complex analysis, which is probably not allowed in competition problems.*

But the idea is interesting and useful, and often not covered in courses on complex analysis, so it is worth looking at. It really only requires knowledge of Cauchy's Residue Theorem.

I shall prove the more general result: Suppose

$$f(z) = \frac{p(z)}{q(z)}$$

is a rational function (where $p(z), q(z)$ are polynomials), with

$$\deg q(z) \ge \deg p(z) + 2;$$

and suppose $f(z)$ has poles at $z_1, \ldots, z_n$, with residues $r_1, \ldots, r_n$ Then

$$\sum_{-\infty}^{\infty} f(n) = 2i\pi^2 \sum_{k=1}^{n} \cot(\pi z_k) r_k.$$

Before proving the theorem, let us apply it to our problem. The function

$$f(z) = \frac{1}{1+z^2} = \frac{i}{2}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

has poles at $z = \pm i$ with residues $\pm i/2$. Since

$$\cot(z) = \frac{\cos z}{\sin z} = i\frac{e^{2iz}+1}{e^{2iz}-1},$$

we have

$$\cot(\pm\pi i) = \mp i\frac{e^{2\pi}+1}{e^{2\pi}-1}.$$

Hence

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+1} = \frac{\pi^2}{2}\frac{e^{2\pi}+1}{e^{2\pi}-1}.$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2+1} &= \frac{1}{2}\left(\sum_{-\infty}^{\infty} \frac{1}{n^2+1} - 1\right) \\ &= \frac{1}{2}\left(\frac{e^{2\pi}+1}{e^{2\pi}-1} - 1\right) \qquad = \frac{1}{e^{2\pi}-1}.\end{aligned}$$

Turning to the theorem, the importance of

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

is that it is periodic with period 1, and has a simple pole with residue $1/\pi$ at $z = 0$, and therefore at each point $z = n \in \mathbb{Z}$. These are the only poles of $\cot(\pi z)$, since

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2} = 0 \iff e^{2\pi z} = 1 \iff z = n \in \mathbb{Z}.$$

It follows that

$$F(z) = \cot(\pi z) f(z)$$

has a simple pole with residue $f(n)/\pi$ at each integer point $z = n$. (It also has poles of course at the poles of $f(z)$.)

Thus if we draw a large circle $\Gamma = \Gamma(N)$ with radius $N+1/2$ about the origin (we add the 1/2 to make sure we miss a pole) then

$$I(N) = \int_{\Gamma(N)} \cot \pi z f(z) dz = \sum_{n=-N}^{N} \frac{f(n)}{\pi} + \Sigma,$$

where

$$\Sigma = 2\pi i \sum_{j=1}^{n} r_i f(z_i) \cot(\pi z_i),$$

summed over the residues of $f(z)$.

It is not difficult to see that $\cot(\pi z)$ is bounded away from the poles, let us say apart from a disk radius 1/4 around each integer point. For

$$\cot(\pi z) = \frac{1 + e^{-2\pi iz}}{1 - e^{-2\pi iz}} = \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1};$$

so

$$\left|e^{\pi iz}\right| \geq 2 \implies \left|e^{-\pi iz}\right| \leq \frac{1}{2} \implies |\cot(\pi z)| \leq \frac{17/4}{3/4} < 6,$$

and similarly

$$\left|e^{-\pi iz}\right| \geq 2 \implies \left|e^{\pi iz}\right| \leq \frac{1}{2} \implies |\cot(\pi z)| \leq \frac{17/4}{3/4} < 6.$$

But if $z = x + iy$ then

$$\left|e^{\pi iz}\right| = e^{-\pi y}$$

so

$$\left|e^{\pi iz}\right| \leq 2 \text{ and } \left|e^{\pi iz}\right| \leq 2 \implies |y| < 1.$$

Thus by the periodicity of $\cot(\pi z)$ we need only consider the compact region

$$\{(x, y) : |x| \leq 1/2,\ |y| \leq 1; x^2 + y^2 \geq \frac{1}{4}\},$$

where $\cot(\pi z)$ *is evidently bounded; so*

$$|\cot(\pi z)| < C$$

provided $|z - n| \geq 1/4$ *for all* $n \in \mathbb{Z}$ *(and in particular on the circles* $\Gamma(N)$*).*

On the other hand, since $\deg q(z) \geq \deg p(z) + 2$ *it follows that*

$$|f(z)| < \frac{C'}{R^2}$$

for sufficiently large z*. Hence*

$$|I(N)| \leq 2\pi(N + 1/2)CC'/(N + 1/2)^2;$$

and so

$$I(N) \to 0 \text{ as } N \to \infty.$$

Thus

$$\sum_{n=-N}^{N} \frac{f(n)}{\pi} \to \Sigma,$$

ie

$$\sum_{-\infty}^{\infty} f(n) = \pi\Sigma.$$

As an addendum, let us see how to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

by this method.

In this case we have to modify the argument slightly, since $f(z) = 1/z^2$ *has a pole at* $z = 0$*. We must compute the residue of*

$$f(z)\cot(\pi z)$$

at $z = 0$*.*

Since

$$\begin{aligned}\cot(\pi z) &= \frac{\cos(\pi z}{\sin(\pi z)} & = \frac{1 - \pi^2 z^2/2! + \cdots}{\pi z - \pi^3 z^3/3! + \cdots} \\ &= \frac{1}{\pi z}\frac{1 - \pi^2 z^2/2! + \cdots}{1 - \pi^2 z^2/3! + \cdots} \\ &= \frac{1}{\pi z}(1 - \pi^2 z^2/2)(1 + \pi^2 z^2/3!) + O(z^4) \\ &= \frac{1}{\pi z}(1 - \pi^2 z^2/3) + O(z^4),\end{aligned}$$

it follows that

$$f(z)\cot(\pi z) = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z).$$

Thus $f(z)\cot(\pi z)$ *has residue* $-\pi/3$ *at* $z = 0$. *Our argument gives*

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2} - \pi/3 = 0,$$

ie

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2/6.$$

10. Determine $\det A$, where A is the $n \times n$ matrix with entries

$$a_{ij} = \frac{1}{x_i + y_j}.$$

Answer: *Let us multiply row* i *by* $\prod_j(x_i + y + j)$, *to give the matrix* B *where*

$$b_{ij} = p_j(x_i),$$

with

$$p_j(x) = \prod_{k \neq j}(x + y_k),$$

and

$$\det A = \frac{1}{\prod)i, j(x_i + y_j)}\det B.$$

Each polynomial $p_j(t)$ is of degree $n-1$ in the variables n^2 variables x_i, y_j; so $\det B$ *is a polynomial of degree $n(n-1)$.*

But if $x_i = x_{i'}$ or ($y_j = y_{j'}$ then $\det B = 0$*. It follows that each of the $n(n-1)$ terms $(x_i - x_{i'})$ and $(y_j - y_{j'})$ is a factor of* $\det B$*. Hence*

$$\det B = c \prod_{i<i'} (x_i - x_{i'}) \prod_{j<j'} (y_j - y_{j'}),$$

for some constant c.

To compute c, note that if we subtract column 1 from each of the other columns we get

$$p_j(x_i) = p_1(x_i) = p_j(x_i) = (y_1 - y_j)$$

11. What is the maximal area of a quadrilateral with sides $1, 2, 3, 4$?

Answer: *Suppose*

$$AB = 1, BC = 2, CD = 3, DA = 4.$$

Suppose

$$D\hat{A}B = \theta, B\hat{C}D = \phi,$$

The areas of the triangles DAB, BCD are

$$\frac{1}{2}|DA|\,|AB| \sin\theta = 2\sin\theta,\ \frac{1}{2}|BC|\,|CD| \sin\phi = 3\sin\phi.$$

Thus the area of the quadrilateral is

$$\Sigma = 2\sin\theta + 3\sin\phi$$

By the cosine rules for these 2 triangles,

$$BD^2 = 4^2 + 1^2 - 8\cos\theta = 2^2 + 3^2 - 12\cos\phi,$$

ie

$$8\cos\theta - 12\cos\phi = 17 - 13,$$

ie

$$2\cos\theta - 3\cos\phi = 1.$$

Squaring and adding,

$$\Sigma^2 + 1^2 = 2^2 + 3^2 + 12(\sin\theta\sin\phi - \cos\theta\cos\phi),$$

ie

$$\Sigma^2 = 12(1 + \cos(\theta + \phi)).$$

Thus Σ *is maximized when*

$$\cos(\theta + \phi) = 1,$$

ie

$$\theta + \phi = \pi,$$

that is, when the quadrilateral is cyclic.

In this case,

$$\Sigma^2 = 24,$$

ie the maximal area is

$$\Sigma = 2\sqrt{6}.$$

12. Can you find an equilateral triangle all of whose vertices have integer coordinates?

    Answer: *Identifying the plane with the field of complex numbers, let the vertices* A, B, C *of the triangle be represented by the complex numbers*

    $$a, b, c \in \Gamma,$$

    the ring of gaussian integers.

    Then

    $$b - a = \omega(c - a),$$

    where $\omega = e^{\pm 2\pi/3}$.

    Thus

    $$\omega = \frac{b-a}{c-a} = x + yi,$$

    where $x, y \in \mathbb{Q}$.

13. Given any two polynomials $f(t), g(t)$, show that there exists a non-zero polynomial $F(x, y)$ such that $F(f(t), g(t)) = 0$ identically.

Answer: *Suppose the degrees of $f(t), g(t)$ are m, n. Take the polynomials*

$$f(t)^i \ (0 \leq i \leq rn), \quad g(t)^j \ (0 \leq j \leq rm),$$

where r is an integer yet to be chosen; and consider the products

$$f(t)^i g(t)^j.$$

There are $r^2 mn$ products, each of degree $< 2rmn$. The polynomials of degree $< 2rmn$ form a vector space of dimension $2rmn$. So if

$$r^2 mn \geq 2rmn,$$

ie

$$r \geq 2,$$

then the products $f(t)^i g(t)^j$ must be linearly dependent over the base field, say

$$\sum a_{i,j} f(t)^i g(t)^j = 0,$$

ie

$$F(f(t), g(t)) = 0,$$

where

$$F(x, y) = \sum_{i,j} a_{ij} x^i y^j.$$

14. Show that the equation

$$y^5 = x^2 + 4$$

has no integer solutions.

Answer: *If p is an odd prime then $(\mathbb{Z}/p)^\times$ is a cyclic group C_{p-1}. It follows that if $5 \mid p - 1$ then there are just $(p -$*

$1)/5$ *'quintic residues'* mod$p$. *For example, there are just 2 such residues* mod11, *and evidently these must be* ± 1.

Accordingly, if there is a solution then

$$x^2 \equiv -3 \text{ or } -5 \bmod 11,$$

ie

$$x^2 \equiv 8 \text{ or } 6 \bmod 11.$$

The quadratic residues mod11 *are*

$$1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 5, 5^2 \equiv 3.$$

So 6 and 8 are both quadratic non-residues, and the equation has no solution.

[The next modulus we could have chosen would be 31, with 6 quintic residues. It is unlikely that none of these is 4 plus one of the 15 quadratic residues.]

15. Show that there exists a real number α such that the fractional part of α^n lies between $1/3$ and $2/3$ for all positive integers n.

Challenge Problem

Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < \left|h\sqrt{m} - k\sqrt{n}\right| < 2\epsilon.$$

Answer: *Although this was a Putnam question, it doesn't seem that difficult. The essential point is that if n is large then $\sqrt{n+1}$ is close to $\sqrt{n}$.*

More precisely, since

$$f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2\sqrt{x}},$$

it follows by the Mean Value Theorem that

$$\sqrt{m+1} - \sqrt{m} = \frac{1}{2\sqrt{m+\theta}},$$

where $0 < \theta < 1$.

Thus if $m \geq M$ *then*

$$\sqrt{M+1} - \sqrt{M} < \frac{1}{2\sqrt{M}}.$$

So if set

$$M^{1/2} \geq \frac{h}{\epsilon}$$

then

$$m \geq M \implies h\sqrt{m+1} - h\sqrt{m} < \epsilon.$$

Now let us choose N *so that*

$$h\sqrt{M} - k\sqrt{N} < 0;$$

and let us successively set $m = M, M+1, M+2, \ldots$.

Then

$$f(m) = h\sqrt{m} - k\sqrt{N}$$

starts negative, and increases by $< \epsilon$ *at each step, but tending ultimately to* ∞. *It follows that*

$$f(m) \in (\epsilon, 2\epsilon)$$

for some m.