Problem Solving (MA2201)

Week 11

Timothy Murphy

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1. Show that if the subset $S \subset \{1, 2, \ldots, 2n\}$ contains more than n numbers then one of these numbers must divide another.

2. Show that the only integral values taken by
$$\frac{x^2 + y^2 + 1}{xy}$$
with integers x, y are ± 3.

3. Find all primes of the form
$$101010 \cdots 101$$
(with alternate digits $0, 1$, beginning and ending with 1).

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property that
$$f(x) - f(y) \leq (x - y)^2$$
for all x, y.

5. Show that the last digit of n^n is periodic. What is the period?

6. For which real numbers x does the sequence
$$x, \cos x, \cos(\cos x), \cos(\cos(\cos x)), \ldots$$
converge?

7. Can you put 6 points on the plane such that the distance between any two is an integer, but no three are collinear?
8. Prove that every integer ≥ 12 is the sum of two composite numbers.
9. How many "minimal paths" are there from one corner of a chessboard to the opposite corner, going along the edges of the squares? (Evidently each minimal path will consist of 16 such edges.)
10. For each real number $r < 0$ the subset $U(r)$ of the complex plane is defined by
$$U(r) = \{z : \left|z^2 + z + 1\right| < r\}.$$
For which r is $U(r)$ connected?
11. Show that
$$\prod_{1 \leq i < j \leq n} \frac{a_i - a_j}{i - j}$$
is an integer for any strictly increasing sequence of integers $a_1, a_2, \ldots, a_n$.
12. What is the largest integer expressible as the product of positive integers with sum 2011?
13. Give two intersecting lines l, m and a constant c, find the locus of a point P such that the sum of the distances from P to the lines m, n is equal to c.
14. The four points A, B, C, D in space have the property that AB, BC, CD, DA touch a sphere at the points P, Q, R, S. Show that P, Q, R, S lie in a plane.
15. How many digits does the number 125^{100} have?

Challenge Problem

The function $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all x. Show that $f(x) = 0$ for all x.