Problem Solving (MA2201)

Week 11

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1. Show that if the subset $S \subset \{1, 2, \ldots, 2n\}$ contains more than n numbers then one of these numbers must divide another.

   Answer: *For each odd number*

   $$a \in \{1, 3, 5, \ldots, 2n-1\}$$

   consider the set of numbers

   $$U(a) = \{a, 2a, 2^2a, \ldots, 2^e a : 2^e a \le 2n < 2^{e+1}a\}.$$

   These subsets $U(1), U(3), \ldots$ *are disjoint, and*

   $$\{1, 2, \ldots, 2n\} = U(1) \cap U(3) \cap \cdots \cap U(2n-1).$$

   There are n subsets; so one of them, say $U(a)$, *must contain two elements of* S:

   $$2^e a, 2^f a \in S.$$

   But then

   $$2^e a \mid 2^f a$$

   if $e < f$.

2. Show that the only integral values taken by

   $$\frac{x^2 + y^2 + 1}{xy}$$

with integers x, y are ± 3.

Answer: *Consider the equation*

$$\frac{x^2 + y^2 + 1}{xy} = k,$$

ie

$$x^2 + y^2 - kxy + 1 = 0$$

for a given integer k.

Suppose (x_0, y_0) is a integer solution of this equation. If we fix y_0, we can regard the equation as a quadratic equation in x. This equation will have a second solution x_1; and since

$$x_0 + x_1 = ky_0,$$

it follows that x_1 is an integer.

Thus (x_0, y_0) gives rise to a second solution (x_1, y_0); and similarly, regarding the equation now as a quadratic in y, this gives rise to a third solution (x_1, y_1), and so on:

$$(x_0, y_0) \to (x_1, y_0) \to (x_1, y_1) \to (x_2, y_1) \to \cdots$$

But now we can use the fact that for two such solutions, say $x_r, y_r), (x_{r+1}, y_r)$ we will have

$$x_r x_{r+1} = -(1 + y_r^2).$$

Note that there is no solution with $k = \pm 1$, Note that there is no solution with $x = \pm y$, since this will give $k = \pm 1$, and then

$$x^2 \pm xy + y^2 = -1,$$

ie

$$(x \pm y/2)^2 + 3y^2/4 = -1,$$

which is absurd.

So we may assume that

$$|y_r| < |x_r|,$$

in which case

$$|x_r x_{r+1}| \leq |y_r|^2,$$

and so

$$|x_{r+1}| < |y_r|.$$

So each solution gives a smaller solution (in the sense that $|x| + |y|$ is smaller), and we would finish with a solution with $x = y = 0$, which we see from the original equation is impossible.

We conclude that there is no solution unless $k = \pm 3$.

3. Find all primes of the form

$$101010 \cdots 101$$

(with alternate digits $0, 1$, beginning and ending with 1).

Answer: *Suppose there are n 1's. Then the number is*

$$\begin{aligned} N &= 100^{n-1} + 100^{n-2} + \cdots + 1 \\ &= \frac{100^n - 1}{99}. &= \frac{(10^n - 1)(10^n + 1)}{99}. \end{aligned}$$

Thus

$$10^n - 1 = rx, 10^n + 1 = sy,$$

where $rs = 99$, and x, y are integers.

If $n > 2$ then one of $x, y > 1$, and since $x, y \mid N$, N cannot be prime.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(x) - f(y) \le (x - y)^2$$

for all x, y.

Answer: *The function $f(x)$ is continuous, since*

$$|f(x) - f(y)| < \epsilon \text{ if } |x - y| < \sqrt{epsilon}.$$

Moreover $f(x)$ is differentiable for all x, with $f'(x) = 0$, since

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \le \lim_{h\to 0} h = 0.$$

Hence $f(x)$ is constant:

$$f(x) = C;$$

and clearly any constant function satisfies the condition.

5. Show that the last digit of n^n is periodic. What is the period?

Answer: *Let us compute*

$$n^n \bmod 5 \text{ and } n^n \bmod 2.$$

It is evident that

$$n^n \equiv \begin{cases} 0 \bmod 2 \text{ if } n \text{ is even} \\ 1 \bmod 2 \text{ if } n \text{ is odd.} \end{cases}$$

In particular

$$(n+2)^{n+2} \equiv n^n \bmod 2$$

for all n.

By Fermat's Little Theorem, if $5 \nmid m$ then

$$m^4 \equiv 1 \bmod 5,$$

and so

$$m^{4r} \equiv 1 \bmod 5.$$

Also

$$(m+5)^n \equiv m^n \bmod 5.$$

and so

$$(m+5s)^n \equiv m^n \bmod 5$$

Putting these together, if $5 \nmid n$ *then*

$$(n+20)^{n+20} \equiv n^{n+20} \equiv n^n \bmod 5.$$

If $5 \nmid n$ *(and* $n > 0$*) then evidently*

$$5 \mid n^n, 5 \mid (n+20)^{n+20};$$

and so

$$(n+20)^{n+20} \equiv 0 \equiv n^{n+20} \bmod 5.$$

Since trivially

$$(n+20)^{n+20} \equiv 0 \equiv n^{n+20} \bmod 2,$$

it follows that

$$(n+20)^{n+20} \equiv 0 \equiv n^{n+20} \bmod 10.$$

Thus $n^n \bmod 10$ *has period dividing 20, ie period 1,2,4,5,10 or 20.*

Clearly the period is not 1,2,4 or 5. (Eg $1^1 \equiv 1$*, while* $6^6 \equiv 6$.*)*

To see that the period is not 10, note that

$$2^2 \equiv 4 \bmod 5 \textit{ while } 12^{12} \equiv 1 \bmod 5$$

since $12^4 \equiv 1$.

Hence the last digit of the sequence n^n *has period 20.*

[As a carry-on problem, show that for any r *the last* r *digits of* n^n *are periodic.]*

6. For which real numbers x does the sequence

$$x, \cos x, \cos(\cos x), \cos(\cos(\cos x)), \ldots$$

converge?

Answer: *Suppose the given sequence*

$$a_{n+1} = \cos(a_n)$$

does converge to ℓ. *Then*

$$\ell = \cos(\ell).$$

Since $|\cos x| \leq 1$, *we must have*

$$-1 \leq \ell \leq 1;$$

and then, since $\cos x > 0$ *for* $-1 \leq x \leq 1$ *we must have*

$$\ell \in [0, 1].$$

Since $\cos x$ *decreases from 1 to* $\cos 1$ *on this interval, while* x *increases from 0 to 1 there is one and only one solution* ℓ *to the equation.*

By the argument above,

$$a_2 = \cos(\cos x)) \in [0, 1],$$

so we only need to consider x *in this range.*

By the Mean Value Theorem,

$$\frac{\cos x - \ell}{x - \ell} = -\sin\theta,$$

where θ *lies between* x *and* ℓ, *and so in* $[0, 1]$.

Setting $x = a_n$, *it follows that*

$$\frac{|a_{n+1} - \ell|}{|a_n - \ell|} \leq \sin 1.$$

Since $\sin 1 < 1$, *we conclude that* a_n *will get closer and closer to* ℓ, *and so*

$$a_n \to \ell$$

for all x.

7. Can you put 6 points on the plane such that the distance between any two is an integer, but no three are collinear?

Answer: *Yes.*

It is sufficient (and simpler) to show that we can find 6 points such that the distance between any two is rational.

Suppose ABC is a rational right-angle triangle with right-angle at B ie a right-angle triangle with sides of rational length). Then it is clear that

$$\sin A, \cos A \in \mathbb{Q}.$$

Conversely, if

$$\sin\theta, \cos\theta \in \mathbb{Q}$$

then θ is the angle in a right-angle triangle.

Consider the reflection ABC' of the triangle in the line BC. The two triangles form a rational isosceles triangle ACC' with $AC = AC'$ and angle $2A$ at A.

Conversely if ACC' is an isosceles triangle, with rational sides $AC = AC'$ and angle 2θ at A, where $\sin\theta, \cos\theta \in \mathbb{Q}$, then the triangle is rational, ie CC' is rational.

Now take points $P_1, P_2, P_3, \ldots, P_n$ equally spaced around a unit circle centre O, with

$$P_1\hat{O}P_2 = P_2\hat{O}P_3 = \cdots = P_{n-1}\hat{O}P_n = 2\theta,$$

where $\sin\theta, \cos\theta \in \mathbb{Q}$.

Note that

$$\sin\theta, \cos\theta \in \mathbb{Q} \implies \sin r\theta, \cos r\theta \in \mathbb{Q}$$

for $r = 1, 2, 3, \ldots,$ since $\sin r\theta, \cos r\theta$ can be expressed as polynomials in $\sin\theta, \cos\theta$ with integral coefficients.

It follows that the distance between any two of our points P_i, P_j is rational, since OP_iP_j is an isosceles triangle with angle 2ϕ, where

$$\phi = (j - i)\theta.$$

We note that the result still holds if the points P_n continue around the circle. So we have in fact constructed an enumerably infinite set of points with rational distance between any two of them.

Strictly speaking, to prove this we should show that the sequence of points does not recur, ie $\theta \neq r\pi$ for any $r \in \mathbb{Q}$.

However, this is not necessary for the question as posed, since we can certainly make the angle θ as small as we like. For we know that we have a pythagorian right-angle triangle with sides $m^2 - n^2, 2mn, m^2 + n^2$, and if we take $m = n + 2$ this has an angle with

$$\sin\theta = \frac{2n+1}{2n^2+2n+1},$$

which can be arbitrarily small.

I shall leave the proposition above as a question for week 12!

8. Prove that every integer ≥ 12 is the sum of two composite numbers.

   Answer: *This is trivial. If n is even then so is $n - 4$, and*

$$n = 4 + (n - 4).$$

   If n is odd then $n - 9$ is even, and

$$n = 9 + (n - 9).$$

9. How many "minimal paths" are there from one corner of a chessboard to the opposite corner, going along the edges of the squares? (Evidently each minimal path will consist of 16 such edges.)

   Answer: *Let us go from the top left corner to the bottom right. Then we must take a sequence of moves like*

$$RRDRDD\ldots,$$

where there are 8 R's and 8 D's, and we have to determine how many ways we can arrange these.

This is the same as the coefficient of x^8 *in*

$$(1+x)(1+x)\cdots(1+x) = (1+x)^{16}.$$

Hence the number of paths is

$$\binom{16}{8}.$$

10. For each real number $r < 0$ the subset $U(r)$ of the complex plane is defined by

$$U(r) = \{z : \left|z^2 + z + 1\right| < r\}.$$

For which r is $U(r)$ connected?

Answer: *Note that*

$$f(z) = z^2 + z + 1 = (z - \omega)(z - \omega^2),$$

where

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

Let $A = \omega, B = \omega^2, C = -1/2$. *Then we can write*

$$U(r) = \{P : PA.PB < r\},$$

where $P = z$.

If $r \leq 0$ *the set* $U(r)$ *is empty. If* $r > 0$ *then*

$$A, B \in U(r).$$

Suppose $P \in U(r)$. *Let* X *be the foot of the perpendicular from* P *to the line* AB. *Then*

$$XA \leq PA, XB \leq PB.$$

So

$$X \in U(r);$$

and the same argument shows that the whole segment

$$PX \subset U(r).$$

Note that since $U(r)$ *is open, it is connected if and only if it is arcwise-connected.*

Suppose $P, Q \in U(r)$*; and suppose* Y *is the foot of the perpendicular from* Q *to the line* AB*. Then by the argument above, if there is a path from* P *to* Q *in* $U(r)$ *then the orthogonal projection of this path onto the line* AB *will give a path on* AB *from* X *to* Y*, and so the segment*

$$XY \subset U(r).$$

In particular A, B *are connected by a path if and only if*

$$AB \subset U(r).$$

Now

$$C \in U(r) \iff r > \frac{3}{4}.$$

Hence U *is not connected if*

$$0 < r \leq \frac{3}{4}.$$

We shall show that if $r > 3/4$ *then* $U(r)$ *is connected. From the argument above it is sufficient to show that the points of the line* AB *in* $U(r)$ *form a connected set.*

To see this, suppose first that Z *lies on the segment* AB*, ie between* A *and* B*. Then by the inequality between the geometric and arithmetic means,*

$$AX.XB \leq (AX + XB)^2/4 = AB^2/4 = 3/4.$$

Hence the segment

$$AB \subset U(r).$$

On the other hand, if X *lies on the line* AB *on the opposite side of* A *to* B *then as* X *moves towards* A *both* XA *and*

XB decrease. Hence if X lies in $U(r)$ so does the segment XA; and similarly if X lies on the other side of B.

Thus if $r > 3/4$ then the points of $U(r)$ on AB form a connected set (ie an open segment), and so the whole of $U(r)$ is connected.

11. Show that

$$\prod_{1\le i<j\le n} \frac{a_i - a_j}{i-j}$$

is an integer for any strictly increasing sequence of integers $a_1, a_2, \ldots, a_n$.

Answer: *Take any prime p. Consider the remainders of the number $a_1, \ldots, a_n \bmod p$. Suppose that $r_0, r_1, \ldots, r_{p-1}$ have remainders $0, 1, \ldots, p-1 \bmod p$ respectively. Then the number of terms $a_i - a_j$ that are divisible by p is*

$$(r_0-1)!(r_1-1)!\ldots(r_{p-1}-1)! \quad (r_0+r_1+\cdots+r_{p-1}=n)$$

The same is true of the terms $i-j$ on the bottom, except that here in place of $r_0, r_1, \ldots, r_{p-1}$ the number of remainders are all either $[n/p]$ or $[n/p]+1$.

To show that the number of terms on the top divisible by p is $\ge$ the corresponding number on the bottom it is sufficient to show that the product

$$(r_0-1)!(r_1-1)!\ldots(r_{p-1}-1)!$$

(subject to the constraint $r_0+r_1+\cdots+r_{p-1}=n$) is minimized when the r_i are as equal as possible. To prove this it is sufficient to show that if two of the r_i differ by 2 or more, then the product is reduced by making them more equal, ie if $r \le r'-2$ then

$$r!r'! > (r+1)!(r'-1)!,$$

ie

$$r' > r+1.$$

By the same argument, To show that the number of terms on the top divisible by p^2 is $\geq$ the number on the bottom divisible by p^2; and the same is true with p^3, p^4, etc. Since the power of p dividing the numerator is equal to the sum of the numbers of terms divisible by $p, p^2, \ldots,$ it follows that the power of p dividing the numerator is $\geq$ the power dividing the bottom.

Since this is true for all primes p, the denominator divides the numerator, so the number is an integer.

12. What is the largest integer expressible as the product of positive integers with sum 2011?

Answer: *Note that if $n \geq 5$ then*

$$2(n-2) > n.$$

It follows that we can increase the product by splitting any factor ≥ 5. So the maximal value will be attained by using only factors 2,3,4.

We can ignore 4 since we can replace any factor 4 by 2^2.

Also

$$3^2 > 2^3,$$

while both have sum 6.

So we can assume that the number of 2's is less that 3.

Hence the maximum will be attained by

$$2^e 3^f,$$

where

$$2e + 3f = 2011,$$

with $0 \leq 3 < 3$.

Since

$$2011 \equiv 1 \bmod 3,$$

we must in fact have 2 2's. So the maximum is

$$2^2 3^{69}.$$

13. Give two intersecting lines l, m and a constant c, find the locus of a point P such that the sum of the distances from P to the lines m, n is equal to c.

Answer: *Recall that the distance of the point* $P = (X, Y)$ *from the line*

$$\ell(x, y) = ax + by + c = 0$$

is

$$d(P, \ell) = \frac{abs\ell(X, Y)}{(a^2 + b^2)^{1/2}}.$$

So the locus will be

$$\pm\frac{\ell(X, Y)}{C_1} \pm \frac{m(X, Y)}{C_2} = c$$

where $C_1, C_2 > 0$. *and we have to choose the signs so that each term is positive. Note that we get a linear form in* X, Y, *ie a line or line-segment, in each case.*

Recall that

$$\ell(X, Y)$$

is positive for points P *on one side of the line, and negative for those on the other. Let us say that the point is at distance* $+d$ *or* $-d$ *from the line, according to the sign.*

It follows that the locus consists of 4 edges of a parallelogram $PQRS$, *where* P *is on the line* ℓ *at distance* $+c$ *from* m, Q *is on the line* m *at distance* $+c$ *from* ℓ, R *is on the line* ℓ *at distance* $-c$ *from* m, *and* S *is on the line* m *at distance* $-c$ *from* ℓ.

14. The four points A, B, C, D in space have the property that AB, BC, CD, DA touch a sphere at the points P, Q, R, S. Show that P, Q, R, S lie in a plane.

Answer:

15. How many digits does the number 125^{100} have?

Answer: *The number is* $200 + r$*, where* r *is the number of digits before the decimal point in* 1.25^{100}*, ie*

$$10^{r-1} < 1.25^{100} < 10^r.$$

Taking logarithms to base 10,

$$r - 1 < 100 \log_{10} 1.25 < r,$$

ie

$$r = [100 \log_{10} 1.25] + 1.$$

Now

$$\begin{aligned} \log_{10} 1.25 &= \frac{\log_e 1.25}{\log_e 10} \\ &= \frac{0.223143551}{2.30258509} \\ &= 0.096910013. \end{aligned}$$

It follows that

$$r = 10,$$

and the number of digits is 210.

Challenge Problem

The function $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all x. Show that $f(x) = 0$ for all x.

Answer: *Suppose* $f(x) \neq 0$ *for some* $x > 0$.

Let x_0 *be the largest integer such that* $f(x) = 0$ *for* $0 \leq x \leq x_0$*; and let* $|f(x)|$ *attain its maximum* M *in* $[x_0.x_0+1/2$ *at* x_0+t*, where* $0 \leq t \leq 1/2$.

By the Mean Value Theorem,

$$\frac{f(x_0 + t)}{t} = f'(t),$$

and so

$$M = t\,|f'(t)|\,.$$

But by hypothesis,

$$|f'(t)| \le |f(t)| \le M,$$

while $0 < t < 1/2$. *Hence*

$$M \le M/2,$$

which is absurd.

Thus $f(x) = 0$ *for all* $x \ge 0$*; and by the same argument applied to* $f(-x)$, $f(x) = 0$ *for all* $x \le 0$. *Hence* $f(x) = 0$ *for all* x.

Comment: *If*

$$\frac{f'(x)}{f(x)} = 1$$

then

$$f(x) = Ce^x;$$

and since $f(0) = 0$ *it follows that* $C = 0$.

We have to translate this idea into something rigorous; and the easiest way to do that is often to use the Mean Value Theorem as we have done.