Problem Solving (MA2201)

Week 10

Timothy Murphy

November 27, 2011

1. Prove that
$$\arctan \sinh t = \arcsin \tanh t.$$

2. Is the set of decreasing sequences of positive integers $n_1 \geq n_2 \geq n_3 \geq \cdots$ enumerable?

3. Show that in any triangle ABC,
$$\sin \frac{A}{2} \leq \frac{a}{b+c}.$$

4. If $a_1, a_2, \ldots a_n$ are distinct natural numbers and none of them is divisible by a prime strictly larger that 3, show that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < 3.$$

5. If three points A, B, C are chosen at random on the circumference of a circle, what is the probability that the centre O of the circle lies inside the triangle ABC?

6. How many incongruent triangles are there with integer sides and perimeter n?

7. Find all solutions in integers of the equation
$$x^2 + 2 = y^3.$$

8. What is the greatest number of parts into which the plane can be divided by n circles?

9. Let (a_n) be a sequence of positive reals such that

$$a_n \leq a_{2n} + a_{2n+1}$$

for all n. Show that $\sum a_n$ diverges.

10. Find all rational numbers a, b, c such that the roots of the equation

$$x^3 + ax^2 + bx + c = 0$$

are just a, b, c.

11. Suppose a, b are coprime positive integers. Show that every integer $n \geq (a-1)(b-1)$ is expressible in the form

$$n = ax + by,$$

with integers $x, y \geq 0$.

12. Can all the vertices of a regular tetrahedron have integer coordinates (m, n, p)?

13. Show that there are infinitely many pairs of positive integers m, n for which $4mn - m - n + 1$ is a perfect square.

14. Each point of the plane is coloured red, green or blue. Must there be a rectangle all of whose vertices are the same colour?

15. Show that a sequence of $mn+1$ distinct real numbers must contain either a sequence of $m+1$ increasing numbers or a sequence of $n+1$ decreasing numbers.

Challenge Problem

The function $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all x. Show that $f(x) = 0$ for all x.