Problem Solving (MA2201)

Week 10

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1. Prove that
$$\arctan \sinh t = \arcsin \tanh t.$$

   Answer:

2. Is the set of decreasing sequences of positive integers $n_1 \geq n_2 \geq n_3 \geq \cdots$ enumerable?

   Answer:

3. Show that in any triangle ABC,
$$\sin \frac{A}{2} \leq \frac{a}{b+c}.$$

   Answer:

4. If $a_1, a_2, \ldots a_n$ are distinct natural numbers and none of them is divisible by a prime strictly larger that 3, show that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < 3.$$

   Answer:

5. If three points A, B, C are chosen at random on the circumference of a circle, what is the probability that the centre O of the circle lies inside the triangle ABC?

   Answer:

6. How many incongruent triangles are there with integer sides and perimeter n?

   Answer:

7. Find all solutions in integers of the equation

$$x^2 + 2 = y^3.$$

Answer: *We know that the ring*

$$\mathbb{Z}[\sqrt{-2}] = \{m + n\sqrt{-2}\}$$

is euclidean, and so has unique factorisation. The only units in this ring are ± 1.

Factorising the left hand side of the equation,

$$x^2 + 2 = (x + \sqrt{-2})(x - \sqrt{-2})$$

Since

$$d \mid x + \sqrt{-2}, x - \sqrt{-2} \implies d \mid 2\sqrt{-2},$$

it follows that

$$\gcd x + \sqrt{-2}, x - \sqrt{-2} = 1, \sqrt{-2}, 2 \text{ or } 2\sqrt{-2}.$$

Taken with the fact that the only units are ± 1, *we see that there are 2 possibilities:*

(a) $x + \sqrt{-2} = u^3, x - \sqrt{-2} = v^3$,

(b) $x + \sqrt{-2} = \sqrt{-2}u^3, x - \sqrt{-2} = \sqrt{-2}v^3$.

Suppose

$$u = a + \sqrt{-2}b, v = a - \sqrt{-2}b.$$

In the first case, we have

$$x + \sqrt{-2} = (a^3 - 6ab^2) + (3a^2b - 2b^3)\sqrt{-2}.$$

Thus

$$b(3a^2 - 2b^2) = 1.$$

It follows that either

$$b = 1, 3a^2 - 2b^2 = 1 \text{ or } b = -1, 3a^2 - 2b^2 = -1,$$

ie

$$b = 1, a = \pm 1 \text{ or } b = -1, 3a^2 = 1.$$

The second choice is impossible; so

$$b = 1, a = \pm 1 \implies x = \pm 5.$$

Similarly, in the second case

$$x + \sqrt{-2} = -2(3a^2b - 2b^3) + (a^3 - 6ab^2)\sqrt{-2},$$

so that

$$a(a^2 - 6b^2) = 1.$$

It follows that either

$$a = 1, a^2 - 6b^2 = 1 \text{ or } a = -1, a^2 - 6b^2 = -1,$$

ie

$$a = 1, b = 0 \text{ or } a = -1, 6b^2 = 2.$$

Again, the second choice is impossible; so

$$a = 1, b = 0 \implies x = 0.$$

We conclude that the only non-trivial solution is

$$x = 5, y = 3.$$

8. What is the greatest number of parts into which the plane can be divided by n circles?

   Answer:

9. Let (a_n) be a sequence of positive reals such that

$$a_n \leq a_{2n} + a_{2n+1}$$

for all n. Show that $\sum a_n$ diverges.

Answer:

10. Find all rational numbers a, b, c such that the roots of the equation

$$x^3 + ax^2 + bx + c = 0$$

are just a, b, c.

Answer: *We have*

$$\begin{aligned} a &= -(a+b+c), \\ b &= ab + ac + bc, \\ c &= -abc. \end{aligned}$$

Thus either $c = 0$ *or*

$$c = -ab.$$

Ignoring the first possibility for the moment, the first equation gives

$$2a + b = ab,$$

ie

$$b(a-1) = 2a,$$

while the second gives

$$b(a-1) = (a+b)ab.$$

Thus

$$2a = (a+b)ab,$$

and so either $a = 0$ *or*

$$a = \frac{2-b}{b}.$$

Ignoring the first possibility, and substituting for b in the second,

$$2a^2 = (a+1)(-a^2+a-1),$$

ie

$$a^3 + 2a^2 - 1 = 0,$$

ie

$$(a+1)(a^2+a-1) = 0.$$

Since a is rational, this implies that

$$a = -1,$$

which is impossible since

$$b(a+1) = 2a.$$

On the other hand, if $c = 0$ then from the original equations,

$$b = -2a \text{ and } b(a-1) = 0.$$

Thus either $b = 0$, in which case

$$a = b = c = 0,$$

or $a = 1$, in which case

$$a = 1, b = -2, c = 0.$$

Hence there are just 3 cubics with the given property:

$$x^3 + x^2 + x + 1,$$

with roots $1, 1, 1$

11. Suppose a, b are coprime positive integers. Show that every integer $n \geq (a-1)(b-1)$ is expressible in the form

$$n = ax + by,$$

with integers $x, y \geq 0$.

Answer: *We know that we can find $x, y \in \mathbb{Z}$ such that*

$$ax + by = 1.$$

It follows that we can find $x, y \in \mathbb{Z}$ such that

$$ax + by = n$$

for any integer n.

It is easy to see that if x_0, y_0 is one solution then the general solution is

$$x = x_0 + bt, \; y = y_0 - at,$$

where $t \in \mathbb{Z}$. In particular there is just one solution with

$$0 \leq x \leq b - 1.$$

But if

$$ax + by \geq (a-1)(b-1) = a(b-1) - b + 1$$

then

$$x \leq b - 1 \implies y > -1 \implies y \geq 0,$$

so that we have a solution with $x, y \geq 0$.

We are not asked this, but it is worth noting that there is no solution of

$$ax + by = ab - a - b = a(b-1) - b$$

with $x, y \geq 0$. For $x = b-1, y = -1$ is the unique solution of this equation with $0 \leq x \leq b-1$; so any solution with $x \geq 0$ satisfies

$$x \geq b - 1 \implies y < 0.$$

12. Can all the vertices of a regular tetrahedron have integer coordinates (m, n, p)?

Answer: *Yes. Take the cube with vertices*

$$(\pm 1, \pm 1, \pm 1),$$

and choose one vertex, say $A = (1, 1, 1)$.

Consider the 3 vertices at distance $2\sqrt{2}$ *from* A*, namely*

$$(1, -1, -1),\ (-1, 1, -1),\ (-1, -1, 1).$$

These are also at distance $2\sqrt{2}$ *from each other; so the 4 vertices form a regular tetrahedron.*

13. Show that there are infinitely many pairs of positive integers m, n for which $4mn - m - n + 1$ is a perfect square.

Answer: *Take*

$$m = t^2, n = 2t^2.$$

We will get a perfect square if

$$5t^2 + 1 = u^2,$$

ie

$$u^2 - 5t^2 = 1,$$

and we know this Pell's equation has an infinity of solutions.

Concretely, $(u, t) = (2, 1)$ *gives*

$$u^2 - 5t^2 = -1$$

ie

$$(2 + \sqrt{5})(2 - \sqrt{5}) = -1.$$

It follows that

$$u + \sqrt{5}t = (2 + \sqrt{5})^2,$$

ie

$$(u, t) = (9, 4)$$

will solve the Pell's equation; and then

$$u + \sqrt{5}t = (9 + 4\sqrt{5})^n \qquad (n = 1, 2, 3, \dots)$$

will give an infinity of solutions.

14. Each point of the plane is coloured red, green or blue. Must there be a rectangle all of whose vertices are the same colour?

    Answer: *Lets consider 2 colours, say red and green, first. We'll only consider rectangles with sides parallel to the axes.*

    Take 3 vertical lines, and consider the colours of the 3 points where a horizontal line meets these 3 lines.

    The colours may be $(R, G, R), (R, G, G)$, etc. There are $2^3 = 8$ possbilities.

    It follows that if there are more than 8 horizontal lines then 2 must have the same colours in the same order.

    Two of these colours must be the same, in the same place, say $(G, R, G), (G, R, G)$. Then the 4 G's are at the vertices of a rectangle.

    Can we extend this to 3 colours?

    Let us take 4 vertical lines. Then the 3 colours can be arranged in $3^4 = 81$ ways. So if we take more than 81 horizontal lines two must have the same colours in the same order. Two of these colours must be the same; so this gives us a rectangle with vertices of this colour.

    Evidently this argument extends to any number of colours.

15. Show that a sequence of $mn+1$ distinct real numbers must contain either a subsequence of $m + 1$ increasing numbers or a subsequence of $n + 1$ decreasing numbers.

    Answer:

Challenge Problem

The function $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all x. Show that $f(x) = 0$ for all x.

Answer: