Problem Solving (MA2201)

Week 1

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1. Show that the product of 3 successive integers is always divisible by 6.
2. What are the last two digits of 2011^{2011}?
3. Given 100 integers $a_1, \ldots, a_{100}$, show that there is a sum of consecutive elements $a_i + \cdots + a_{i+j}$ divisible by 100.
4. Show that if $\cos a = b$ and $\cos b = a$ then $b = a$.
5. Show that there are an infinite number of positive integers n such that $4n$ consists of the same digits in reverse order.
6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the identity $f(f(x)) = x$ holds for all real x.
7. Prove that
   $$x^y + y^x > 1$$
   for all positive real x, y.
8. The rectangle $ABCD$ has sides $AB = 1, BC = 2$. What is the minimum of $AE + BE + EF + CF + DF$ for any two points E, F in $ABCD$?
9. The 8 numbers $x_1, x_2, \ldots, x_8$ have the property that the sum of any three consecutive numbers is 16. If $x_2 = 9$ and $x_6 = 2$, what are the values of the remaining numbers?

10. Can you find a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f'(x) = f(x+1)$$

such that $f(x) \to \infty$ as $x \to \infty$?

11. The set of pairs of positive reals (x, y) such that

$$x^y = y^x$$

form the straight line $y = x$ and a curve. Determine the point at which the curve cuts the line.

12. Show that there is just one tetrahedron whose edges are consecutive positive integers and whose volume is a positive integer.

13. Every point of the circle is colored using one of two colors — black or white. Show that there exists some isosceles triangle that has vertices on that circle and which has all its vertices the same color).

14. Given 11 integers $x_1, ..., x_{11}$ show that there must exist some non-zero finite sequence $a_1, ..., a_{11}$ of elements from $\{-1, 0, 1\}$ such that the sum $a_1x_1 + \cdots + a_{11}x_{11}$ is divisible by 2011.

15. I have two children, one of whom is a boy born on a Tuesday. What is the probability that my other child is a boy?

Challenge Problem[1]

The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all $x, y \in \mathbb{R}$. Show that $f(x) = 0$ for all $x \leq 0$.

[1] Each week one very difficult problem will be posed. This one took me 8 days to solve!