# Problem Solving (MA2201)

# Week 1

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1. Show that the product of 3 successive integers is always divisible by 6.

   **Answer:** *One (or two) of the numbers is divisible by 2, and one of the numbers is divisible by 3. Since* $\gcd(2,3) = 1$ *it follows that the product of the numbers is divisible by* $2 \cdot 3$.

2. What are the last two digits of $2011^{2011}$?

   **Answer:**

   **Method 1** *We have to determine*

   $$2011^{2011} \bmod 100.$$

   *First of all, since*

   $$2011 \equiv 11 mod 100$$

   *it follows that*

   $$2011^{2011} \equiv 11^{2011} \bmod 100.$$

   *By Fermat's Last Theorem (or the extension to it), if* $a$ *is coprime to* $n$ *then*

   $$a^{\phi(n)} \equiv 1 \bmod n,$$

*where $\phi()$ is Euler's 'totient function' (the number of numbers between 0 and $n$ coprime to $n$).*
*This function is 'multiplicative in the number-theoretic sense, ie*

$$\gcd(m,n) = 1 \implies \phi(mn) = \phi(m)\phi(n).$$

*Thus*

$$\phi(100) = \phi(2^2)\phi(5^2).$$

*But if $p$ is prime,*

$$\phi(p^e) = p^e - p^{e-1} = p^{e-1}(p-1).$$

*(This follows easily, since a number is not coprime to $p^e$ if and and only if it is divisible by $p$.)*
*Hence*

$$\phi(100) = (2 \cdot 1)(5 \cdot 4) = 40.$$

*So*

$$11^{40} \equiv 1 \bmod 100.$$

*It follows that*

$$11^{2011} \equiv 11^{11} \bmod 100.$$

*Now*

$$11^2 = 121 \equiv 21 \bmod 100, \quad (1)$$
$$11^3 \equiv 11 \cdot 21 \equiv 31 \bmod 100, \quad (2)$$

*and so on, until*

$$11^9 \equiv 91 \bmod 100, \quad (3)$$
$$11^{10} \equiv 11 \cdot 91 \equiv 1 \bmod 100, \quad (4)$$
$$(5)$$

*and finally,*

$$11^{11} \equiv 11 \bmod 100.$$

**Method 2** *We can use the binomial theorem instead of Fermat's Last Theorem in this case:*

$$11^{11} = (1+10)^{11} \tag{6}$$
$$= 1 + 10 \cdot 11 + \binom{11}{2} 10^2 + \cdots \equiv 1 + 110 \bmod 100, \tag{7}$$

*since all the terms after the first 2 are divisible by 100. Thus*

$$11^{11} \equiv 11 \bmod 100.$$

3. Given 100 integers $a_1, \dots, a_{100}$, show that there is a sum of consecutive elements $a_i + \cdots + a_{i+j}$ divisible by 100.

   **Answer:** *A classic Pigeon Hole Principle problem.*

   *Let*

   $$s_i = a_1 + a_2 + \cdots + a_i,$$

   *with* $s_0 = 0$.

   *Then two of the 101 integers* $s_i$ *must have the same remainder* mod100, *say*

   $$s_i \equiv s_j \bmod 100,$$

   *where* $i < j$.

   *But then*

   $$s_j - s_i \equiv 0 \bmod 100,$$

   *ie*

   $$a_{i+1} + a_{i+2} + \cdot + a_j \equiv 0 \bmod 100/$$

4. Show that if $\cos a = b$ and $\cos b = a$ then $b = a$.

   **Answer:** *Let*

   $$f(x) = \cos(\cos x).$$

*Then*

$$f(a) = a \text{ and } f(b) = b.$$

*Since* $-1 \leq \cos x \leq 1$, $\cos(-x) = \cos(x)$, *and* $\cos x$ *is decreasing from 1 to* $\cos(1) as x runs from 0 to 1$, *it follows that*

$$\cos(\cos(1)) = f(1) \leq f(x) \leq \cos 1.$$

*Thus we need only consider* $x$ *in the range*

$$[f(1), \cos 1] \subset (0, \pi/2).$$

*Since* $\cos x$ *is decreasing in this range,* $f(x) = \cos(\cos x)$ *is also decreasing. Hence there is at most one point where* $f(x) = x$. *It follows that*

$$a = b.$$

5. Show that there are an infinite number of positive integers $n$ such that $4n$ consists of the same digits in reverse order.

   **Answer:** *Suppose*

$$\begin{aligned} n &= ab \ldots cd \\ 4n &= dc \ldots ba. \end{aligned}$$

*Then* $a = 1$ *or* $2$, *since otherwise* $4n$ *would have more digits than* $n$.

*On the other hand* $4n$ *must end with an even digit. Hence* $a = 2$.

*Since* $4a = 8$ *we must have* $d = 8$ *or* $9$. *But if* $d = 9$ *as last digit of* $n$, *then* $a = 6$ *as the last digit of* $4n$. *Hence* $d = 8$.

*Although it is not necessary in this case, let us try to formalize the situation.*

*Suppose we have determined the first $r-1$ digits of $n$ and $4n$, and so the last $r-1$ digits also. We are trying to find the $r$th digits $(u, U)$ of $n, 4n$.*

*From the first $r-1$ digits, we know the number $e$ which must be carried over from the $r$th digit; and similarly from the last $r-1$ digits, we know the number $f$ which is carried over to the new digit. Let us also denote by $E$ the number (unknown at present) which is carried over to the $(r+1)$th digit; and by $F$ the number on the right that will be carried over. We know that $e, E, f, F \in \{0, 1, 2, 3\}$.*

*Now*

$$4u + E = 10e + U$$

*on the left, while*

$$4U + f = 10F + u$$

*on the right.*

*Let us write this*

$$[e, E] \to (u, U) \to [f, F].$$

*At the first (and last) digits, we have*

$$[0, 0] \to (2, 8) \to [0, 3].$$

*At the second digit, $[e, f] = [0, 3]$, and so*

$$4u + E = U,\ 4U + 3 = 10F + u.$$

*From the first equation, $4u \leq 9$ and so $u \in \{0, 1, 2\}$.*

*From the second equation $u$ is odd. Hence*

$$u = 1.$$

*Now from the first equation,* $U \in \{4, 5, 6, 7\}$*; while from the second equation,*

$$4U + 3 \equiv 1 \bmod 10.$$

*Hence* $U \in \{2, 7\}$.

*Thus*

$$[0, 3] \to (1, 7) \to [3, 3].$$

*Next we start with* $[e, f] = [3, 3]$. *We have*

$$4u + E = 30 + U,\ 4U + 3 = 10F + u.$$

*From the first equation,* $u \in \{7, 8, 9\}$*; while from the second equation,* $u$ *is odd. Thus* $u \in \{7, 9\}$.

*If* $u = 7$*, then from the first equation* $U \in \{0, 1\}$*, while from the second equation*

$$4U + 3 \equiv 7 \bmod 10,$$

*and so* $U \in \{1, 6\}$. *Thus in this case,*

$$[3, 3] \to (7, 1) \to [3, 0].$$

*On the other hand, if* $u = 9$*, then from the first equation* $U \in \{6, 7, 8, 9\}$*, while from the second equation*

$$4U + 3 \equiv 9 \bmod 10,$$

*and so* $U \in \{4, 9\}$. *Thus in this case,*

$$[3, 3] \to (9, 9) \to [3, 3].$$

*Next we start with* $[e, f] = [3, 0]$. *We have*

$$4u + E = 30 + U,\ 4U = 10F + u.$$

*From the first equation,* $u \in \{7, 8, 9\}$*; while from the second equation,* $u$ *is even. Thus* $u = 8$.

*Now from the first equation* $U \in \{2, 3, 4, 5\}$*, while from the second equation*

$$4U \equiv 8 \bmod 10,$$

*and so* $U \in \{2, 7\}$. *Thus in this case,*

$$[3, 0] \to (8, 7) \to [2, 0].$$

6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the identity $f(f(x)) = x$ holds for all real $x$.

7. Prove that

$$x^y + y^x > 1$$

for all positive real $x, y$.

**Answer:** *Let*

$$f(x, y) = x^y + y^x.$$

*If* $x \geq 1$ *then* $x^y \geq 1$. *Similarly, if* $y \geq 1$ *then* $y^x \geq 1$. *So we can limit ourselves to the region* $0 < x < 1,\ 0 < y < 1$.

*Suppose* $f(x, y) < 1$ *at some point in this region.*

*At a minimum,*

$$\frac{\delta f}{\delta x} = 0 \text{ and } \frac{\delta f}{\delta y} = 0.$$

*But*

$$\frac{\delta f}{\delta x} = x^{y-1} + xy^x,$$
$$\frac{\delta f}{\delta y} = yx^y + y^{x-1}.$$

*Thus*

$$x^y + x^2y^x = 9,\ y^2x^y + y^x = 0.$$

*Hence*

$$1 - x^2y^2 = 0.$$

*But* $x^2y^2 < 1$ *in the region. Hence the function does not have a minimum, and so*

$$f(x, y) > 1$$

*for all* $x, y > 0$

8. The rectangle $ABCD$ has sides $AB = 1, BC = 2$. What is the minimum of $AE + BE + EF + CF + DF$ for any two points $E, F$ in $ABCD$?

   **Answer:** *Since* $AE+BE+EF+CF+DF$ *is a continuous function of* $E$ *and* $F$*, 2e know that the minimum is attained at some points* $E, F$ *in the square.*

   *Keeping* $F$ *fixed, it follows that*

   $$AE + BE + FE$$

   *is a minimum at this point.*

9. The 8 numbers $x_1, x_2, \ldots, x_8$ have the property that the sum of any three consecutive numbers is 16. If $x_2 = 9$ and $x_6 = 2$, what are the values of the remaining numbers?

   **Answer:** *This is trivial.*

   *Since* $x_2 = 9$*,*

   $$9 + x_3 + x_4 = 16.$$

   *Hence*

   $$x_3 + x_4 = 7.$$

   *But*

   $$x_3 + x_4 + x_5 = 16.$$

   *Hence*

   $$x_5 = 9.$$

*Since*

$$x_4 + x_5 + x_6 = 16,$$

*it follows that*

$$x_4 = 16 - (9 + 2) = 5,$$

*and so*

$$x_3 = 16 - (x_4 + x_5) = 16 - (5 + 9) = 2.$$

*Hence*

$$x_1 = 16 - (x_2 + x_3) = 16 - (9 + 2) = 5.$$

*Finally,*

$$x_7 = 16 - (x_5 + x_6) = 16 - (9 + 2) = 5,$$

*and*

$$x_8 = 16 - (x_6 + x_7) = 16 - (2 + 5) = 9.$$

*So the 8 numbers are: 5, 9, 2, 5, 9, 2, 6, 9.*

10. Can you find a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f'(x) = f(x + 1)$$

such that $f(x) \to \infty$ as $x \to \infty$?

**Answer:** *There is no such function. For if $f(x) \to \infty$ then $f(x) > 0$ for $x \geq c$. Hence $f'(x) = f(x + 1) > 0$ for $x \geq c$. In particular $f(x)$ is increasing in this range.*

*Suppose $x \geq c$. By the Mean Value Theorem,*

$$f(x + 1) - f(x) = f'(x + \theta) = f(x + \theta + 1)$$

*for some $\theta \in (0, 1)$.*

*But then*

$$f(x + \theta + 1) < f(x + 1),$$

*contradicting the fact that $f(x)$ is increasing.*

11. The set of pairs of positive reals $(x, y)$ such that

$$x^y = y^x$$

form the straight line $y = x$ and a curve. Determine the point at which the curve cuts the line.

**Answer:**

12. Show that there is just one tetrahedron whose edges are consecutive positive integers and whose volume is a positive integer.

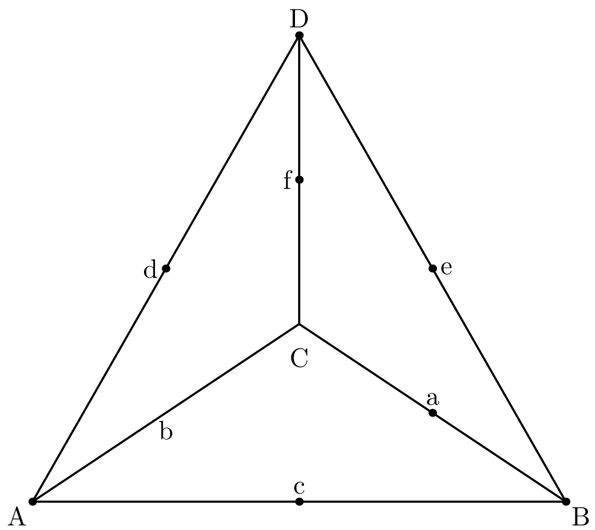
**Answer:** *I haven't been able to answer this question, so the following remarks are comments rather than proofs.*

*tetrahedron*

(a) *There is a formula for the volume of a tetrahedron in terms of the lengths of the sides. See* `<www.cs.berkeley.edu/~wkahan/VtetLang.pdf>`.

*Suppose the tetrahedron is $ABCD$. Let us denote the sides by*

$$a = BC,\ b = CA,\ c = AB,\ d = DA,\ e = DB,\ f = DC.$$



*Let $d, e, f$ also denote (by 'abuse of notation' as the French say) the vectors $DA, DB, DC$. Then the volume*

$$V = \frac{1}{6}\det X,$$

*where $X = (d, e, f)$, ie $X$ is the matrix with columns $d, e, f$.*
*Thus*

$$\begin{aligned} 36V^2 &= \det X'X \\ &= \det \begin{pmatrix} d.d & d.e & d.f \\ e.d & e.e & e.f \\ f.d & f.e & f.f \end{pmatrix} \end{aligned}$$

*Consider the triangle $DAB$. By the 'cosine rule' for triangles,*

$$\begin{aligned} d.e &= |d|\,|e|\,cos(A\hat{D}B) \\ &= (d^2 + e^2 - c^2)/2, \end{aligned}$$

*Thus we have expressed $V^2$ in terms of the lengths of the sides:*

$$288V^2 = \det \begin{pmatrix} 2d^2 & d^2+e^2-c^2 & d^2+f^2-b^2 \\ d^2+e^2-c^2 & 2e^2 & e^2+f^2-a^2 \\ d^2+f^2-b^2 & e^2+f^2-a^2 & 2f^2 \end{pmatrix}$$

*(b) Masha pointed out that there are 'trivial' tetrahedron with zero volume, whose edges are consecutive positive integers, eg*

$$a = 1,\ b = 3,\ c = 4,\ d = 2,\ e = 5,\ f = 6.$$

*In fact in this case all four 'faces' have zero area.*

(c) *To find the volume of a regular tetrahedron with unit side, consider the cube with vertices* $(\pm 1, \pm 1, \pm 1)$. *The 4 points* $(1,1,1),\ (1,-1,-1),\ (-1,1,-1),\ (-1,-1,1)$ *are distance* $2\sqrt{2}$ *apart, and so form a regular tetrahedron with side* $2\sqrt{2}$ *and volume*

$$\frac{1}{6}\det\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} = \frac{8}{3}.$$

*It follows that a regular tetrahedron with side 1 has volume*

$$\frac{8}{3(2\sqrt{2})^3} = \frac{\sqrt{2}}{12}.$$

(d) *There is another version of the formula for the volume, based on the 3 pairs* $(a, d),\ (b, e),\ (c, f)$ *of opposite edges:*

$$\begin{aligned}(12V)^2 =& (a^2 + d^2)(-a^2d^2 + b^2e^2 + c^2f^2) \\ &+ (b^2 + e^2)(a^2d^2 - b^2e^2 + c^2f^2) \\ &+ (c^2 + f^2)(a^2d^2 + b^2e^2 - c^2f^2) \\ &- a^2b^2c^2 - a^2e^2f^2 - b^2d^2f^2 - c^2d^2e^2,\end{aligned}$$

*where the 4 terms in the last line correspond to the faces of the tetrahedron. (*`<http://math.arizona.edu/~eacosta/pdfs/docs/Master.pdf>`*)*

*We can permute the 6 edges in* $6!$ *ways, of which* $4!$ *arise from permutation of the 4 vertices. It follows that there are just* $6!/4! = 30$ *different ways of assigning 6 different lengths to the 6 edges, if we regard 2 ways as the same whenever one can be derived from the other by re-naming the vertices.*

*Evidently any of the* $3! = 6$ *permutations of the 3 edge-pairs can be brought about by a permutation of the 4*

*vertices. It follows that* $4!/3! = 4$ *permutations of the vertices must send each edge-pair into itself. One of these is the identity; and it is easy to see that each of the others will send the edges in one edge-pair into themselves, and will swap the edges in each of the two other edge-pairs. (For example, the permutation* $(AD)(BC)$ *of the vertices sends each of the edges* $a, d$ *into themselves, but swaps the edges* $b, e$ *and the edges* $c, f$*.)*
*We know that*

$$a, b, c, d, e, f = 0, 1, 2, 3, 4, 5 \bmod 6$$

*in some order.*
*Let's consider the equation modulo 3 first. Two of the lengths are congruent to* $0, 3 \bmod 6$*. Suppose they are in different arm-pairs. We may suppose without loss of generality that*

$$b \equiv c \equiv 0 \bmod 3,$$

*while*

$$a^2, d^2, e^2, f^2 \equiv 1 \bmod 3.$$

*This gives*

$$(12V)^2 \equiv 2 \cdot -1 + 1 \cdot 1 + 1 \cdot 1 - 1 = -1 \bmod 3,$$

*which is impossible. It follows that that the 2 lengths* $\equiv 0, 3 \bmod 6$ *must belong to the same arm-pair. We may assume that these are*

$$a \equiv 0 \bmod 6, d \equiv 3 \bmod 6.$$

*Now consider the equation modulo 4. Just two of* $b, c, e, f$ *are even. Suppose they belong to the same arm-pair. We may assume that*

$$b, e \equiv 0 \bmod 2.$$

*Then*

$$a^2, b^2, e^2 \equiv 0 \bmod 4 \textit{ while } d^2, c^2, f^2 \equiv 1 \bmod 4.$$

*This gives*

$$(12V)^2 \equiv 1 \cdot 1 + 2 \cdot -1 - 1 = 2 \bmod 4,$$

*which is impossible. It follows that each arm-pair contains one arm of even length and one of odd length.*
*The six edges have lengths*

$$6n+r, 6n+r+1, 6n+r+2, 6n+r+3, 6n+r+4, 6n+r+5,$$

*where $r$ is one of* $0, 1, 2, 3, 4, 5$.
*Suppose first that $r = 0$, ie the shortest edge has length divisible by 6. Then*

$$a = 6n.d = 6n + 3.$$

*We may assume that*

$$b = 6n + 1, e = 6n + 4.$$

*This leaves 2 choices:*

$$(c, f) = (6n + 2, 6n + 5) \textit{ or } (6n + 5, 6n + 2).$$

13. Every point of the circle is colored using one of two colors — black or white. Show that there exists some isosceles triangle that has vertices on that circle and which has all its vertices the same color).

    **Answer:**

14. Given 11 integers $x_1, ..., x_{11}$ show that there must exist some non-zero finite sequence $a_1, ..., a_{11}$ of elements from $\{-1, 0, 1\}$ such that the sum $a_1x_1 + \cdots + a_{11}x_{11}$ is divisible by 2011.

**Answer:** *Another application of the Pigeon Hole Principle. Consider the* $2^{11} = 2048$ *numbers*

$$a_1x_1 + \cdots + a_{11}x_{11}$$

*where* $a_i = 0$ *or* $1$.

15. I have two children, one of whom is a boy born on a Tuesday. What is the probability that my other child is a boy?

## Challenge Problem[1]

**The function** $f : \mathbb{R} \to \mathbb{R}$ **satisfies**

$$f(x + y) \leq yf(x) + f(f(x))$$

**for all** $x, y \in \mathbb{R}$**. Show that** $f(x) = 0$ **for all** $x \leq 0$**.**

[1] Each week one very difficult problem will be posed. This one took me 8 days to solve!