Problem Solving (MA2201)

Reading Week Micro-Project

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If you are taking this course, you should choose ONE of the projects/problems below (taken mainly from V I Arnold's "Trivium" document), and complete a short — one- or two-page — report, hopefully solving the problem, but also outlining the ideas or theory behind your answer.

Add a short list of online or other references that helped you with the project.

1. Find the Betti numbers of the following surfaces in 3-dimensional projective space:
   (a) $x^2 + y^2 = 1 + z^2$,
   (b) $z = xy$,
   (c) $z = x^2$,
   (d) $z^2 = x^2 + y^2$.
2. Find the critical values and critical points of the map $z \mapsto z^2 + \bar{z}$. (Illustrate your reply.)
3. Find the flux of the vector field $\vec{r}/r^3$ across the surface
$$(x-1)^2 + y^2 + z^2 = 2.$$
4. Find the sum of the indices of the singular points of the vector field $z\bar{z}^2 + z^4 + \bar{z}^4$.
5. Find the index of the singular point at the origin of the vector field
$$(x^4 + y^4 + z^4, x^3y - yx^3, xyz^2).$$

6. Find the index of the singular point at the origin of the vector field
$$\operatorname{grad}(xy + yz + zx).$$

7. Calculate the integral of the gauss curvature of the surface
$$x^4 + (x^2 + y^2 - 1)(2x^2 + 3y^2 - 1) = 0.$$

8. Find the self-intersection index of the surface $x^4 + y^4$ in the projective plane $\mathbb{CP}^2$.

9. Study the topology of the Riemann surface of the function
$$w = \arctan z.$$

10. How many handles does the Riemann surface
$$w = \sqrt{1 + z^n}$$
possess?

11. Find the number of positive and negative squares in the canonical forms of the quadratic forms
$$\sum_{i<j} (x_i - x_j)^2$$
and
$$\sum_{i<j} x_i x_j.$$

12. Decompose the space of homogeneous polynomials of degree 5 in x, y, z as a sum of irreducible subspaces invariant under the rotation groups $\mathbf{SO}(3)$.

13. Alice secretly holds a €10 or €20 note in her hand, and Bob guesses what she is holding. If Bob is right he takes the money, but if he is wrong he pays Alice €15. Is the game fair? And what is the best strategy for each player?

14. Find the average of the solid angle of the disk x^2+y^2 in the plane $z = 0$ as seen from the points of the sphere $x^2 + y^2 + (z - 2)^2 = 1$.

15. Do the medians of a triangle in the Lobatchevsky plane meet? Do the altitudes meet?

16. Find the Betti numbers of the surface

$$x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_l^2 = 1$$

and of the set

$$x_1^2 + \cdots + x_k^2 \leq 1 + y_1^2 + \cdots + y_l^2$$

in a vector space of dimension $k + l$.

17. Find the eigenvalues (with multiplicities) of the Laplacian on a sphere of radius R in n-dimensional euclidean space.

18. Find a conformal transformation of the unit disk into the first quadrant.

19. Find the Green's function of the operator $d^2/dx^2 - 1$ and solve the equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} u(y) dy = e^{-x^2}.$$

20. Determine the subgroups $S \subset S_4$ up to conjugacy, and for each conjugacy class of subgroups find a polynomial $p(x) \in \mathbb{Z}[x]$ having S as galois group. [You may find the program GAP useful.]

21. An infinite wire lattice is formed by joining each pair of neighbouring vertices in the integer lattice (m, n) with a resistance of 1 ohm. What is the resistance of the lattice between neighbouring vertices?