IMC 2013, Blagoevgrad, Bulgaria

Day 2, August 9, 2013

Problem 1. Let z be a complex number with $|z+1|>2$. Prove that $|z^3+1|>1$.

(10 points)

Problem 2. Let p and q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1}(-1)^{\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor}=\begin{cases}0 & \text{if } pq \text{ is even,}\\ 1 & \text{if } pq \text{ is odd.}\end{cases}$$

(Here $\lfloor x\rfloor$ denotes the integer part of x.)

(10 points)

Problem 3. Suppose that $v_1,\dots,v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unit vector u such that

$$|u\cdot v_i|\le 1/\sqrt{d}$$

for $i=1,2,\dots,d$.

(Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.)

(10 points)

Problem 4. Does there exist an infinite set M consisting of positive integers such that for any $a,b\in M$, with $a<b$, the sum $a+b$ is square-free?

(A positive integer is called square-free if no perfect square greater than 1 divides it.)

(10 points)

Problem 5. Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called *good*, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

(Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.)

(10 points)