AMM problems August–September 2013 due before 31 October, 2013

TCDmath Problem Group
Trinity College, Dublin, Ireland *

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MM-11712 In the game of *Bulgarian solitaire*, n identical coins are distributed into two piles, and a *move* takes one coin from each existing pile to form a new pile. Beginning with a single pile of size n, how many moves are needed to reach a position on a cycle (a position that will eventually repeat)? For example, $5 \mapsto 41 \mapsto 32 \mapsto 221 \mapsto 311 \mapsto 32$, so the answer is 2 when $n = 5$.

MM-11713 Let $x_1, \ldots, x_n$ be nonnegative real numbers. Let $S = \sum_{k=1}^n x_k$. Prove that

$$\prod_{k=1}^n (1 + x_k) \le 1 + \sum_{k=1}^n \left(1 - \frac{k}{2n}\right)^{k-1} \frac{S^k}{k!}.$$

MM-11714 Let $ABCD$ be a cyclic quadrilateral (the four vertices lie on a circle). Let $e = |AC|$ and $f = |BD|$. Let r_a be the inradius of BCD, and define r_b, r_c, r_d similarly. Prove that $er_ar_c = fr_br_d$.

MM-11715 Prove that

$$\sum_{k=0}^\infty \frac{1}{(6k+1)^5} = \frac{1}{2}\left(\frac{2^5-1}{2^5} \cdot \frac{3^5-1}{3^5}\zeta(5) + \frac{11}{8}\left(\frac{\pi}{3}\right)^5 \cdot \frac{1}{\sqrt{3}}\right).$$

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MM-11716 Let $\alpha = (\sqrt{5}-1)/2$. Let p_n and q_n be the numerator and denominator of the nth continued fraction convergent to α. (Thus, p_n is the nth Fibonacci number and $q_n = p_{n+1}$.) Show that

$$\sqrt{5}\left(\alpha - \frac{p_n}{q_n}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{q_n^{2k+2} 5^k},$$

where C_k denotes the kth *Catalan number*, given by $C_k = \frac{(2k)!}{k!(k+1)!}$.

MM-11717 Given a circle c and a line segment AB tangent to c at a point E that lies strictly between A and B, provide a compass and straightedge construction of the circle through A and B to which c is internally tangent.

MM-11718 Given positive real numbers $a_1, \ldots, a_n$ with $n \geq 2$, minimize $\sum_{i=1}^{n} x_i$ subject to the conditions that $x_1, \ldots, x_n$ are positive and that $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i x_i$.