Course 2316 — Sample Paper 3

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The exam will last for 2 hours.
Attempt 5 questions. All carry the same mark.

1. Show that
$$\sum_{p \text{ prime}} \frac{1}{p}$$
is divergent.

2. How many numbers between 1 and 1 million are not divisible by any of the 10 integers $1 - 10$?

3. State (without proof) the Prime Number Theorem.

   Show that the theorem implies that
$$p_n \sim n \log n,$$
where p_n is the nth prime.

4. Find all the generators of the multiplicative group $(\mathbb{Z}/23)^\times$.

   Is the group $(\mathbb{Z}/25)^\times$ (formed by the invertible elements of $\mathbb{Z}/(25)$) cyclic? If so, find a generator.

5. Show that if $2^m + 1$ is prime then $m = 2^n$ for some $n \in \mathbb{N}$.

   Show that the Fermat number
$$F_n = 2^{2^n} + 1,$$
where $n > 0$, is prime if and only if
$$3^{2^{2^n - 1}} \equiv -1 \bmod F_n.$$

6. Suppose

$$n - 1 = 2^e m,$$

where m is odd. Show that if n is prime, and a is coprime to n, then either

$$a^m \equiv 1 \bmod n$$

or else

$$2^f a^m \equiv -1 \bmod n$$

for some $f \in [0, e)$.

Show conversely that if this is true for all a coprime to n then n is prime.

7. State without proof Gauss' Quadratic Reciprocity Law.

Does there exist a number n such that n^2 ends in the digits 1234? If so, find the smallest such n.

8. What is meant by an *algebraic number* and by an *algebraic integer*?

Show that the algebraic integers in the field $\mathbb{Q}(\sqrt{-3}$ form the ring $\mathbb{Z}[\omega]$, where $\omega = (1 + \sqrt{-3})/2/$

Show that this ring is a unique factorisation domain, and determine the units and primes in this domain.