Course 2316 — Sample Paper 2

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Attempt 5 questions. All carry the same mark.

1. Determine $d = \gcd(2009, 2317)$, and find integers m, n such that
$$2009m + 2317n = d.$$

2. Find the smallest positive multiple of 2009 ending in the digits 001, or else show that there is no such multiple.

3. Define Euler's totient function $\phi(n)$, and show that if a is coprime to n then
$$a^{\phi(n)} \equiv 1 \bmod n.$$
Determine the smallest power of 2317 ending in the digits 001.

4. Explain what is meant by a *primitive root* modulo an odd prime p. and find all primitive roots mod 19.

5. Show that if $d > 0$ is not a perfect square then Pell's equation
$$x^2 - dy^2 = 1$$
has an infinity of integer solutions.

   Does the equation
$$x^2 - 5y^2 = -1$$
have an integer solution?

6. Express each of the following numbers as a sum of two squares, or else show that the number cannot be expressed in this way:
$$23,\ 101,\ 2009,\ 2010,\ 2317.$$

7. Show that if the prime p satisfies $p \equiv 3 \bmod 4$ then

$$M = 2^p - 1$$

is prime if and only if

$$\phi^{2^p} \equiv -1 \bmod M,$$

where $\phi = (\sqrt{5} + 1)/2$.

8. Show that the ring $\mathbb{Z}[\sqrt{2}]$ formed by the numbers $m + n\sqrt{2}$ $(m, n \in \mathbb{Z})$ is a Unique Factorisation Domain, and determine the units and primes in this domain.