Chapter 17

Continued fractions

17.1 Finite continued fractions

Definition 17.1. *A* finite continued fraction *is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}},$$

where $a_i \in \mathbb{Z}$ with $a_1, \dots, a_n \geq 1$. We denote this fraction by

$$[a_0, a_1, \dots, a_n].$$

Example: The continued fraction

$$[2, 1, 3, 2] = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

represents the rational number

$$\begin{aligned} 2 + \cfrac{1}{1 + \cfrac{2}{7}} &= 2 + \frac{7}{9} \\ &= \frac{25}{9}. \end{aligned}$$

Conversely, suppose we start with a rational number, say

$$\frac{57}{33}.$$

To convert this to a continued fraction:

$$\frac{57}{33} = 1 + \frac{14}{33}.$$

Now invert the remainder:

$$\frac{33}{14} = 2 + \frac{5}{14}.$$

Again:

$$\frac{14}{5} = 2 + \frac{4}{5},$$

and again:

$$\frac{5}{4} = 1 + \frac{1}{4},$$

and finally:

$$\frac{4}{1} = 4.$$

Thus

$$\frac{57}{33} = [1, 2, 2, 1, 4].$$

Note that the numbers $1, 2, 2, 1, 4$ in the continued fraction are precisely the quotients that would arise if we used the Euclidean Algorithm to compute $\gcd(57, 33)$.

We can consider continued fractions — particularly when we come to *infinite* continued fractions — as a generalisation or extension of the Euclidean Algorithm.

17.2 The p's and q's

We can consider

$$[a_0, a_1, a_2, \ldots, a_n]$$

as a *function* of the variables $a_0, a_1, \ldots, a_n$. Evidently

$$[a_0, a_1, a_2, \ldots, a_n] = \frac{P}{Q},$$

where P, Q are polynomials in $a_0, a_1, \ldots, a_n$ with integer coefficients. This does not define P, Q precisely; but we shall give a precise *recursive* definition below, using induction on the *length* n of the continued fraction.

We start with the continued fraction

$$[a_0] = a_0 = \frac{a_0}{1},$$

setting

$$p = a_0, \ q = 1,$$

Now suppose that we have defined p, q for continued fractions of length $< n$; and suppose that under this definition

$$\alpha_1 = [a_1, a_2, \ldots, a_n] = \frac{p'}{q'}.$$

Then

$$\begin{aligned}\alpha &= a_0 + \frac{1}{\alpha_1}\\ &= a_0 + \frac{q'}{p'}\\ &= \frac{a_0 p' + q'}{p'}.\end{aligned}$$

So we set

$$p = a_0 p' + q', \ q = p'$$

as the definition of p, q for a continued fraction of length n. We set this out formally in

Definition 17.2. *The 'canonical representation' of a continued fraction*

$$[a_0, a_1, a_2, \ldots, a_n] = \frac{p}{q}$$

is defined by induction on n, setting

$$p = a_0 p' + q', \ q = p',$$

where

$$[a_1, a_2, \ldots, a_n] = \frac{p'}{q'}$$

is the canonical representation for a continued fraction of length $n-1$. The induction is started by setting

$$[a_0] = \frac{a_0}{1}.$$

Henceforth if we write

$$[a_0, a_1, a_2, \ldots, a_n] = \frac{p}{q},$$

then p, q will refer to the canonical representation defined above.

17.3 Successive approximants

Definition 17.3. *If*

$$\alpha = [a_0, a_1, \ldots, a_n]$$

then we call

$$[a_0, a_1, \ldots, a_i] = \frac{p_i}{q_i}$$

the ith convergent *or* approximant *to α (for $0 \le i \le n$).*

Example: Continuing the previous example, the successive approximants to

$$\frac{57}{33} = [1, 2, 2, 1, 4]$$

are

$$\begin{aligned}
\frac{p_0}{q_0} &= [1] = \frac{1}{1},\\
\frac{p_1}{q_1} &= [1,2] = 1 + \frac{1}{2} = \frac{3}{2},\\
\frac{p_2}{q_2} &= [1,2,2] = [1,5/2] = 1 + \frac{2}{5} = \frac{5}{7},\\
\frac{p_3}{q_3} &= [1,2,2,1] = [1,2,3] = [1,7/3] = 1 + \frac{3}{7} = \frac{10}{7},\\
\frac{p_4}{q_4} &= [1,2,2,1,4] = [1,2,2,5/4] = [1,2,14/5] = [1,33/14] = \frac{57}{33}.
\end{aligned}$$

Note that while we normally assume that the entries a_n in continued fractions are integers (with $a_n \geq 1$ for $n \geq 1$), it makes sense to use fractional (or even variable) entries, using our recursive formulae for p_n, q_n as functions of $a_0, a_1 \dots$. Usually this will only involve the last entry, where

$$[a_0, \dots, a_{n-1}, a_n, x] = [a_0, \dots, a_{n-1}, a_n + 1/x].$$

Note too that
$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \frac{p_3}{q_3} < \frac{p_1}{q_1};$$
first we get the even convergents, increasing, and then the odd convergents, in reverse order, with the actual number sandwiched in between.

As we shall see, this is the general situation; moreover, the successive convergents are very good approximants to the given number.

Theorem 17.1. *If*
$$\alpha = [a_0, a_1, \dots, a_n]$$
then

$$\begin{aligned}
p_i &= a_i p_{i-1} + p_{i-2},\\
q_i &= a_i q_{i-1} + q_{i-2},
\end{aligned}$$

for $i = 2, 3, \dots, n$.

Proof. We argue by induction on n.

The result follows by induction for $i \neq n$, since the convergents involved are — or can be regarded as — convergents to

$$[a_0, a_1, \dots, a_{n-1}],$$

covered by our inductive hypothesis.

It remains to prove the result for $i = n$. In this case, by the inductive definition of p, q,

$$\begin{aligned}
p_n &= a_0 p'_{n-1} + q'_{n-1},\\
p_{n-1} &= a_0 p'_{n-2} + q'_{n-2},\\
p_{n-2} &= a_0 p'_{n-3} + q'_{n-3}.
\end{aligned}$$

But now by our inductive hypothesis,

$$p'_{n-1} = a_n p'_{n-2} + p'_{n-3}, q'_{n-1} \qquad = a_n q'_{n-2} + q'_{n-3},$$

since

$$a'_{n-1} = a_n,$$

ie the $(n-1)$th entry in α' is the nth entry in α.

Hence

$$\begin{aligned}
p_n &= a_0 p'_{n-1} + q'_{n-1}, \\
&= a_0(a_n p'_{n-2} + p'_{n-3}) + (a_n q'_{n-2} + q'_{n-3}), \\
&= a_n(a_0 p'_{n-2} + q'_{n-2}) + (a_0 p'_{n-3} + q'_{n-3}), \\
&= a_n p_{n-1} + p_{n-2};
\end{aligned}$$

with the second result

$$q_n = a_n q_{n-1} + q_{n-2}$$

following in exactly the same way. □

We can regard this as a recursive definition of $\frac{p_i}{q_i}$, starting with

$$\frac{p_0}{q_0} = \frac{a_0}{1}, \ \frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1},$$

and defining

$$\frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}, \dots$$

successively.

Actually, we can go back two futher steps.

Proposition 17.1. *If we set*

$$\begin{aligned}
p_{-2} = 1, \ q_{-2} = 0, \\
p_{-1} = 0, \ q_{-1} = 1,
\end{aligned}$$

then

$$\begin{aligned}
p_i &= a_i p_{i-1} + p_{i-2}, \\
q_i &= a_i q_{i-1} + q_{i-2},
\end{aligned}$$

for all $i \ge 0$.

One more or less obvious result.

Proposition 17.2. *Both the p's and q's are strictly increasing:*

$$\begin{aligned}
0 < q_0 < q_1 \cdots < q_n, \\
p_0 < p_1 \cdots < p_n.
\end{aligned}$$

Proof. This follows at once by repeated application of the recursive identities

$$p_i = a_i p_{i-1} + p_{i-2}, q_i = a_i q_{i-1} + q_{i-2},$$

since $a_1, a_2, \dots, a_n > 0$ and $q_0 = 1, \ q_1 = a_1,$. □

17.4 Uniqueness

Consider the continued fraction for a rational number x. If $n > 0$ and $a_n > 1$ then

$$[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1].$$

And if $n = 0$, ie $x \in \mathbb{Z}$, then

$$[x_0] = [x_0 - 1, 1].$$

Thus with our example above,

$$\frac{57}{33} = [1, 2, 2, 1, 4] = [1, 2, 2, 1, 3, 1].$$

So there are at least 2 ways of expressing x as a continued fraction.

Proposition 17.3. *A rational number $x \in \mathbb{Q}$ has just two representations as a continued fraction: one with $n = 0$ or $n > 1$, $a_n > 1$, and the other with $n > 0$ and $a_n = 1$.*

Proof. It is sufficient to show that x has just one representation of the first kind. Suppose

$$x = [a_0, a_1, \ldots, a_m] = [b_0, b_1, \ldots, b_n],$$

We may assume that $m \leq n$.

We argue by induction on n. The result is trivial if $m = n = 0$.

Lemma 17.1. *If $n > 0$ and $a_n > 1$ then*

$$a_0 < [a_0, a_1, a_2, \ldots, a_n] < a_0 + 1.$$

Proof. We argue, as usual, by induction on n. This tells us that

$$[a_1, a_2, \ldots, a_n] > 1,$$

from which the result follows, since

$$[a_0, a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{[a_1, a_2, \ldots, a_n]}.$$

□

It follows that

$$[x] = a_0 = b_0.$$

Thus

$$x - a_0 = \frac{1}{[a_1, a_2, \ldots, a_m]} = \frac{1}{[b_1, b_2, \ldots, b_n]} \implies [a_1, a_2, \ldots, a_m] = [b_1, b_2, \ldots, b_n],$$

from which the result follows by induction. □

We will take the first form for the continued fraction of a rational number as standard, ie we shall assume that the last entry $a_n > 1$ unless the contrary is stated.

17.5 A fundamental identity

Theorem 17.2. *Successive convergents $p_i/q_i, p_{i+1}/q_{i+1}$ to the continued fraction $[a_0, a_1, \ldots, a_n]$ satisfy the identity*

$$p_i q_{i+1} - q_i p_{i+1} = (-1)^{i+1}.$$

Proof. We argue by induction on i, using the relations

$$\begin{aligned} p_i &= a_i p_{i-1} + p_{i-2}, \\ q_i &= a_i q_{i-1} + q_{i-2}. \end{aligned}$$

Thus

$$\begin{aligned} p_i q_{i+1} - q_i p_{i+1} &= p_i(a_{i+1} q_i + q_{i-1}) = q_i(a_{i+1} p_i + p_{i-1}) \\ &= p_i q_{i-1} - q_i p_{i-1} \\ &= -(p_{i-1} q_i - q_{i-1} p_i) \\ &= -(-1)^i \\ &= (-1)^{i+1}. \end{aligned}$$

The result holds for $i = -2$ since

$$\begin{aligned} p_{-2} q_{-1} - q_{-2} p_{-1} &= 0 \cdot 0 - 1 \cdot 1 \\ &= (-1)^{-1}. \end{aligned}$$

We conclude that the result holds for all $i \geq 0$. ☐

Proposition 17.4. *The even convergents are monotonically increasing, while the odd convergents are monotonically decreasing:*

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots \leq x \leq \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Proof. By the last Proposition, if i is even then

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} = \frac{p_{i+1} q_i - p_i q_{i+1}}{q_i q_{i+1}} = \frac{1}{q_i q_{i+1}}.$$

Thus

$$\frac{p_i}{q_i} < \frac{p_{i+1}}{q_{i+1}}.$$

Moreover,

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_{i+2}}{q_{i+2}} = \frac{p_{i+1} q_{i+2} - p_{i+2} q_{i+1}}{q_{i+2} q_{i+1}} = \frac{1}{q_{i+2} q_{i+1}} < \frac{1}{q_{i+2} q_{i+1}}.$$

It follows that p_{i+2}/q_{i+2} is closer than p_i/q_i to p_{i+1}/q_{i+1}. Hence

$$\frac{p_i}{q_i} < \frac{p_{i+2}}{q_{i+2}} < \frac{p_{i+1}}{q_{i+1}}.$$

So the even convergents are increasing; and similarly the odd convergents are decreasing.

Also, any even convergent is less than any odd convergent; for if i is even and j is odd then

$$\frac{p_i}{q_i} < \frac{p_{i+j-1}}{q_{i+j-1}} < \frac{p_{i+j}}{q_{i+j}} < \frac{p_j}{q_j}.$$

And since x is equal to the last convergent, it must be sandwiched between the even and odd convergents. ☐

17.6 Infinite continued fractions

So far we have been considering continued fraction expansions of *rational* numbers. But the concept extends to any *real* number $\alpha \in \mathbb{R}$.

Suppose α is irrational. We set

$$a_0 = [\alpha],$$

and let

$$\alpha_1 = \frac{1}{\alpha - a_0}.$$

Then we define $a_1, a_2, \ldots,$ successively, setting

$$\begin{aligned} a_1 &= [\alpha_1], \\ \alpha_2 &= \frac{1}{\alpha_1 - a_1}, \\ a_2 &= [\alpha_2], \\ \alpha_3 &= \frac{1}{\alpha_2 - a_2}, \end{aligned}$$

and so on.

Proposition 17.5. *Suppose*

$$a_0, a_1, a_2, \dots \in \mathbb{Z} \text{ with } a_1, a_2, \dots > 0.$$

Let

$$[a_0, a_1, \ldots, a_i] = \frac{p_i}{q_i}.$$

Then the sequence of convergents converges:

$$\frac{p_i}{q_i} \to x \text{ as } i \to \infty.$$

Proof. It follows from the finite case that the even convergents are increasing, and the odd convergents are decreasing, with the former bounded by the latter, and conversely:

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \dots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

It follows that the even convergents must converge, to α say, and the odd convergents must also converge, to β say.

But if i is even,

$$\frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} = \frac{1}{q_i q_{i+1}}.$$

Since

$$\frac{p_i}{q_i} < \alpha \le \beta < \frac{p_{i+1}}{q_{i+1}},$$

it follows that

$$0 \le \beta - \alpha < \frac{1}{q_i q_{i+1}} < \frac{1}{q_i^2}.$$

Hence

$$\alpha = \beta,$$

ie the convergents tend to a limit $\alpha \in \mathbb{R}$. ∎

Proposition 17.6. *Each irrational number $\alpha \in \mathbb{R}$ has a unique expression as an infinite continued fraction*

$$\alpha = [a_0, a_1, a_2, \ldots].$$

Proof. One could argue that this follows from the algorithm above for constructing the continued fractions of α.

Express each rational numbers $x < \alpha$ as a continued fraction. For simplicity, let us choose the version with final entry $a_n > 1$.

Lemma 17.2. *Suppose*

$$\alpha = [a_0, a_1, \ldots,], \ \beta = [b_0, b_1, \ldots,];$$

and suppose

$$a_0 = b_0, \ldots, a_{n-1} = b_{n-1}, a_n < b_n.$$

Then

$$\alpha < \beta \text{ if } n \text{ is even,}$$
$$\alpha > \beta \text{ if } n \text{ is odd.}$$

Proof. This follows easily from the fact that even convergents are increasing, odd convergents decreasing. □

Now let a_0 be the largest first entry among rational $x < \alpha$; let a_1 be the least second entry among those rationals with a_0 as first entry; let a_2 be the largest third entry among those rationals with a_0, a_1 as first two entries; and so on. Then it is a simple exercise to show that

$$\alpha = [a_0, a_1, a_2, dots].$$

(Note that if the a_n (with given $a_0, \ldots, a_{n-1}$) at the $(n+1)$th stage were unbounded then it would follow that α is rational, since

$$[a_0, \ldots, a_{n-1}, x] \to [a_0, \ldots, a_{n-1}]$$

if $x \to \infty$.)

□

17.7 Diophantine approximation

Theorem 17.3. *If p_n/q_n is a convergent to $\alpha = [a_0, a_1, a_2, \ldots]$ then*

$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{q_n^2}.$$

Proof. Recall that α lies between successive convergents $p_n/q_n, \ p_{n+1}/q_{n+1}$. Hence

$$\begin{aligned} |\alpha - \frac{p_n}{q_n}| &\leq |\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}| \\ &= \frac{1}{q_n q_{n+1}} \\ &\leq \frac{1}{q_n^2}. \end{aligned}$$

□

Remarks:

1. There is in fact inequality in the theorem except in the very special case where α is rational, p_n/q_n is the last but one convergent, and $a_{n+1} = 1$; for except in this case $q_n < q_{n+1}$.

2. Since
$$\frac{1}{q_n q_{n+1}} = \frac{1}{q_n(a_n q_n + q_{n-1})} \leq \frac{1}{a_n q_n^2},$$
if $a_n > 1$ then
$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{2q_n^2}.$$
In particular, if α is irrational then there are an infinity of convergents satisfying
$$|\alpha - \frac{p}{q}| \leq \frac{1}{2q^2}$$
unless $a_n = 1$ for all $n \geq N$.

   In this case
$$\begin{aligned}\alpha &= [a_0, a_1, \ldots, a_n, \phi] \\ &= \frac{p_n\phi + p_{n-1}}{q_n\phi + q_{n-1}} \\ &\in \mathbb{Q}(\sqrt{5}).\end{aligned}$$

We have seen that the convergents are good approximations to α. The next result shows that, conversely, good approximations are necessarily convergents.

Theorem 17.4. *If*
$$|\alpha - \frac{p}{q}| \leq \frac{1}{2q^2}$$
then p/q is a convergent to α.

Proof. Let us express p/q as a continued fraction:
$$\frac{p}{q} = \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

We want to express α in the form
$$\alpha = [a_0, \ldots, a_n, x] = [a_0, \ldots, a_n + \frac{1}{x}].$$

In this case
$$\alpha = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}.$$

Solving for x,
$$\begin{aligned}x &= -\frac{q_n\alpha - p_n}{q_{n-1}\alpha - p_{n-1}} \\ &= -\frac{\alpha - p_n/q_n}{\alpha - p_{n-1}/q_{n-1}}\end{aligned}$$

We want to ensure that $x > 0$. This will be the case if

$$(\alpha - \frac{p_n}{q_n}) \text{ and } (\alpha - \frac{p_{n-1}}{q_{n-1}})$$

are of opposite sign, ie α lies between the two convergents.

At first this seems a matter of good or bad luck. But recall that there are two ways of representing p/q as a continued fraction, one of even length and one odd. (One has last entry $a_n > 1$, and the other has last entry 1.)

We can at least ensure in this way that α lies on the same side of p_n/q_n as p_{n-1}/q_{n-1}, since even convergents are $<$ odd convergents; so if $\alpha > p/q$ then we choose n to be even, while if $\alpha < p/q$ we choose n to be odd.

This ensures that $x > 0$. Now we must show that $x \geq 1$; for then if

$$x = [b_0, b_1, b_2, \ldots]$$

we have

$$\alpha = [a_0, \ldots, a_n, b_0, b_1, b_2, \ldots],$$

and

$$\frac{p}{q} = [a_0, \ldots, a_n]$$

is a convergent to α, as required.

But now

$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{2q_n^2};$$

and since

$$|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}| = \frac{1}{q_n q_{n-1}}$$

it follows that

$$\begin{aligned}
|\alpha - \frac{p_{n-1}}{q_{n-1}}| &\geq |\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}| - |\alpha - \frac{p_n}{q_n}| \\
&\geq \frac{1}{q_n q_{n-1}} - \frac{1}{2q_n^2} \\
&\geq \frac{1}{q_n^2} - \frac{1}{2q_n^2} \\
&= \frac{1}{2q_n^2},
\end{aligned}$$

and so

$$|x| = \frac{|\alpha - p_n/q_n|}{|\alpha - p_{n-1}/q_{n-1}|} \geq 1.$$

□

17.8 Quadratic surds and periodic continued fractions

Recall that a quadratic surd is an irrational number of the form

$$\alpha = x + y\sqrt{d},$$

where $x, y \in \mathbb{Q}$, and $d > 1$ is square-free. In other words,

$$\alpha \in \mathbb{Q}(\sqrt{d}) \setminus \mathbb{Q}$$

for some quadratic field $\mathbb{Q}(\sqrt{d})$.

Theorem 17.5. *The continued fraction of $\alpha \in \mathbb{R}$ is periodic if and only if x is a quadratic surd.*

Proof. Suppose first that α has periodic continued fraction:

$$\alpha = [a_0, \dots, a_m, b_0, \dots, b_n, b_0, \dots, b_n, \dots].$$

Let

$$\begin{aligned} \beta &= [b_0, \dots, b_n, \beta] \\ &= \frac{\beta p'_n + p'_{n-1}}{\beta q'_n + q'_{n-1}} \end{aligned}$$

be the purely periodic part. Then β satisfies the quadratic equation

$$p'_{n-1}\beta^2 + (q'_{n-1} - p'_n)\beta - q'_n = 0,$$

and so is a quadratic surd. And since

$$\begin{aligned} \alpha &= [a_0, \dots, a_m, \beta] \\ &= \frac{\beta p_m + p_{m-1}}{\beta q_m + q_{m-1}}, \end{aligned}$$

it too is a quadratic surd.

The converse is more difficult. Suppose

$$\alpha = [a_0, a_1, \dots]$$

satisfies the quadratic equation

$$F(x) \equiv Ax^2 + 2Bx + C = 0 \quad (A, B, C \in \mathbb{Z}).$$

Let

$$\alpha_n = [a_n, a_{n+1}, \dots].$$

We have to show that

$$\alpha_{m+n} = \alpha_n$$

for some $m, n \in \mathbb{N},\ m > 0$.

We shall do this by showing that α_n satisfies a quadratic equation with bounded coefficients.

Writing θ for a_{n+1}, for simplicity,

$$\begin{aligned} \alpha &= [a_0, \dots, a_n, \theta] \\ &= \frac{\theta p_n + p_{n-1}}{\theta q_n + q_{n-1}}. \end{aligned}$$

Thus

$$A(\theta p_n + p_{n-1})^2 + 2B(\theta p_n + p_{n-1})(\theta q_n + q_{n-1}) + C(\theta q_n + q_{n-1})^2 = 0,$$

ie

$$A'\theta^2 + 2B'\theta + C',$$

where

$$\begin{aligned}
A' &= Ap_n^2 + 2Bp_nq_n + Cq_n^2, \\
B' &= Ap_np_{n-1} + 2B(p_nq_{n-1} + p_{n-1}q_n) + Cq_nq_{n-1}, \\
C' &= Ap_{n-1}^2 + 2Bp_{n-1}q_{n-1} + Cq_{n-1}^2.
\end{aligned}$$

Now

$$A' = q_n^2 F(p_n/q_n).$$

Since $F(\alpha) = 0$ and p_n/q_n is close to α, $F(p_n/q_n)$ is small.

More precisely, since

$$|\alpha - \frac{p_n}{q_n}| \le \frac{1}{q_n^2},$$

it follows by the Mean Value Theorem that

$$\begin{aligned}
F(p_n/q_n) &= -(F(\alpha) - F(p_n/q_n)) \\
&= -F'(t)(\alpha - p_n/q_n),
\end{aligned}$$

where $t \in [\alpha, \alpha + p_n/q_n]$.

Thus if we set

$$M = \max_{t\in[\alpha-1,\alpha+1]} |F'(t)|$$

then

$$|F(p_n/q_n)| \le \frac{M}{q_n^2}$$

and so

$$|A'| \le M.$$

Similarly

$$|C'| \le M.$$

Much the same argument applies to

$$B' = q_nq_{n-1}F^+(p_n/q_n, p_{n-1}/q_{n-1},$$

where

$$F^+(x, y) = Axy + B(x + y) + C$$

is the 'polarised' form of the quadratic form $F(x)$.

Note that

$$F(x^2) - F^+(x, y) = (x - y)(Ax + B) = \frac{1}{2}(x - y)F'(x).$$

Hence

$$F(p_n/q_n) - F^+(p_n/q_n, p_{n-1}/q_{n-1}) = \frac{1}{2}(p_n/q_n - p_{n-1}/q_{n-1})F'(p_n/q_n),$$

and so

$$|F(p_n/q_n) - F^+(p_n/q_n, p_{n-1}/q_{n-1})| \le \frac{M}{2q_nq_{n-1}}.$$

Since

$$|F(p_n/q_n)| \le \frac{M}{q_n^2} < \frac{M}{q_nq_{n-1}},$$

we conclude that

$$|B'| = q_nq_{n-1}|F^+(p_n/q_n, p_{n-1}/q_{n-1}| \le \frac{3}{2}M.$$

Thus A', B', C' are bounded for all n. We conclude that one (at least) of these equations occurs infinitely often; and so one of the α_n occurs infinitely often, ie α is periodic. $\square$

Example: Let us determine the continued fraction for $\sqrt{3}$. We have

$$\begin{aligned}
\sqrt{3} &= 1 + (\sqrt{3} - 1),\\
\frac{1}{\sqrt{3}-1} &= \frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2},\\
\frac{2}{\sqrt{3}-1} &= \sqrt{3} + 1 = 2 + (\sqrt{3}-1),\\
\frac{1}{\sqrt{3}-1} &= 1 + \frac{\sqrt{3}-1}{2},\\
&\dots
\end{aligned}$$

Thus

$$\sqrt{3} = [1, \overline{1, 2}],$$

where we have overlined the periodic part.