Exercise 12

In exercises 1-5, determine the gcd of the given gaussian integers

** 1. $\gcd(3+2i, 2+3i)$

** 2. $\gcd(12, 9-3i)$

** 3. $\gcd(5-5i, 3+i)$

** 4. $\gcd(99, 17)$

** 5. $\gcd(13+2i, 7-11i)$

In exercises 6-10, factorise the given gaussian integer into (gaussian) primes.

** 6. $3+5i$

** 7. $5+3i$

*** 8. $23+17i$

** 9. $11+2i$

** 10. $29-i$

In exercises 11-15, either express the given number as a sum of two squares, or else show that this is not possibles.

** 11. 233

** 12. 317

** 13. 613

** 14. 1009

** 15. 2010

*** 16. Find a formula expressing

$$(x^2+y^2+z^2+t^2)(X^2+Y^2+Z^2+T^2)$$

as a sum of 4 squares.

*** 17. Show that every prime p can be expressed as a sum of 4 squares.

** 18. Deduce from the last 2 exercises that every $n \in \mathbb{N}$ can be expressed as a sum of 4 squares.

** 19. Show that if $n \equiv 7 \bmod 8$ then n cannot be expressed as a sum of 3 squares.

*** 20. Show that if $n = 4^e(8m+7)$ then n cannot be expressed as a sum of 3 squares.

*** 21. Suppose $p \equiv 1 \bmod 4$ is prime. If $p = m^2 + n^2$, find $u, v \in \mathbb{N}$ such that

$$2p = u^2 + v^2,$$

and show that this representation of $2p$ as a sum of 2 squares is unique.

**** 22. Show that if p is a prime such that

$$2p = n^2 + 1$$

then p is the sum of the squares of two consecutive integers.

*** 23. Show that if the prime $p = m^2 + n^2$ and $p \equiv \pm 1 \bmod 10$ then

$$5 \mid xy.$$

*** 24. Find the smallest $n \in \mathbb{N}$ such that $n, n+1, n+2$ are each a sum of 2 squares, but none is a perfect square.

**** 25. Show that there are arbitrarily long gaps between successive integers expressible as a sum of 2 squares.