Exercise 1

In exercises 1–3 determine the gcd d of the given numbers m, n and find integers x, y such that $d = mx + my$.

* 1. $23, 39$

* 2. $87, -144$

* 3. $2317, 2009$.

** 4. Given integers $m, n > 0$ with $\gcd(m, n) = 1$ show that all integers $N \geq mn$ are expressible in the form

$$N = mx + ny$$

with $x, y \geq 0$.

** 5. Find the greatest integer n *not* expressible in the form

$$n = 17x + 23y$$

with $x, y \geq 0$.

*** 6. Which integers n are *not* expressible in the form

$$n = 17x - 23y$$

with $x, y \geq 0$?

** 7. Define the gcd

$$d = \gcd(n_1, n_2, \ldots, n_r)$$

of a finite set of integers $n_1, n_2, \ldots, n_r \in \mathbb{Z}$; and show that there exist integers $x_1, x_2, \ldots, x_r \in \mathbb{Z}$ such that

$$n_1x_1 + n_2x_2 + \cdots + n_rx_r = d.$$

* 8. Find $x, y, z \in \mathbb{Z}$ such that

$$24x + 30y + 45z = 1.$$

*** 9. How many ways are there of paying €10 in 1,2 and 5 cent pieces?

** 10. Show that if $m, n > 0$ then

$$\gcd(m, n) \times \mathrm{lcm}(m, n) = mn.$$

*** 11. Show that if $m, n > 0$ then

$$\gcd(m + n, mn) = \gcd(m, n).$$

** 12. Show that if $n \geq 9$ and both $n-2$ and $n+2$ are prime then $3 \mid n$.

*** 13. Suppose

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Show that $f(n)$ cannot be a prime for all n unless $f(x)$ is constant.

*** 14. Find all integers $m, n > 1$ such that

$$m^n = n^m.$$

*** 15. If $p^e \parallel n!$ show that

$$e = [n/p] + [n/p^2] + [n/p^3] + \cdots .$$

[Note: if p is a prime we say that p^e *exactly divides* N, and we write $p \parallel N$ if $p^e \mid N$ but $p^{e+1} \nmid N$.]

*** 16. How many zeros does 1000! end with?

*** 17. Prove that $n!$ divides the product of any n successive integers.

*** 18. If F_n is the nth Fibonacci number, show that

$$\gcd(F_n, F_{n+1}) = 1$$

and

$$\gcd(F_n, F_{n+2}) = 1.$$

[Note: $F_0 = 1, F_1 = 2$ and $F_{n+2} = F_n + F_{n+1}$.]

** 19. Use the program `/usr/games/primes` on the mathematics computer system to find the next 10 primes after 1 million. [You can find how to use this program by giving the command `man primes`.]

** 20. Use the program `/usr/games/factor` on the mathematics computer system to factorise 123456789. [You can find how to use this program by giving the command `man factor`.]

** 21. Show that the product of two successive integers cannot be a perfect square.

*** 22. Can the product of three successive integers be a perfect square?

***** 23. Show that there are an infinity of integers $x, y, z > 1$ such that

$$x^x y^y = z^z.$$