Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over \( \mathbb{C} \).

1. What is a group representation? What is meant by saying that a representation is simple? What is meant by saying that a representation is semisimple? Prove that every finite-dimensional representation \( \alpha \) of a finite group over \( \mathbb{C} \) is semisimple.

2. Show that all simple representations of an abelian group are of degree 1. Determine from first principles all simple representations of \( D(6) \).

3. Determine all groups of order 30.

4. Prove that the number of simple representations of a finite group \( G \) is equal to the number of conjugacy classes in \( G \).

More questions overleaf!
Show that if the finite group $G$ has simple representations $\sigma_1, \ldots, \sigma_s$ then
\[ \deg^2 \sigma_1 + \cdots + \deg^2 \sigma_s = |G|. \]

Determine the degrees of the simple representations of $S_6$.

5. Determine the conjugacy classes in $A_5$, and draw up the character table of this group.

6. If $\alpha$ is a representation of the finite group $G$ and $\beta$ is a representation of the finite group $H$, define the representation $\alpha \times \beta$ of the product-group $G \times H$.

Show that if $\alpha$ and $\beta$ are simple then so is $\alpha \times \beta$, and show that every simple representation of $G \times H$ is of this form.

Show that the symmetry group $G$ of a cube is isomorphic to $C_2 \times S_4$.

Into how many simple parts does the permutation representation of $G$ defined by its action on the vertices of the cube divide?

7. Show that every representation of a compact group is semisimple.

Determine the simple representations of $U(1)$.

Verify that the simple characters of $U(1)$ are orthogonal.

8. Show that $SU(2)$ has just one simple representation of each degree $0, 1, 2, \ldots$.

Determine the simple representations of $U(2)$.

9. Show that $SU(2)$ is isomorphic to $Sp(1)$ (the group of unit quaternions). Define a 2-fold covering $Sp(1) \rightarrow SO(3)$, and so determine the simple representations of $SO(3)$.