# Course 3413 — Group Representations

# Sample Paper I

Dr Timothy Murphy

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*Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.*

1. What is a *group representation*?

   What is meant by saying that a representation is *simple*?

   Determine from first principles all simple representations of $S(3)$.

2. What is meant by saying that the representation $\alpha$ is *semisimple*?

   Prove that every finite-dimensional representation $\alpha$ of a finite group over $\mathbb{C}$ is semisimple.

   Show that the natural representation of $S_n$ in $\mathbb{C}^n$ (by permutation of coordinates) splits into 2 simple parts, for any $n > 1$.

3. Determine the conjugacy classes in $S_4$, and draw up its character table.

   Determine also the representation-ring for $S_4$, ie express the product $\alpha\beta$ of each pair of simple representations as a sum of simple representations.

*More questions overleaf!*

4. Prove that the number of simple representations of a finite group $G$ is equal to the number of conjugacy classes in $G$.

5. Show that if the finite group $G$ has simple representations $\sigma_1, \ldots, \sigma_s$ then

$$\deg^2 \sigma_1 + \cdots + \deg^2 \sigma_s = |G|.$$

   Determine the degrees of the simple representations of $S_6$.

6. Show that the dodecahedron has 60 proper symmetries, and determine how these are divided into conjugacy classes.

7. Define a *measure* $\mu$ on a compact space $X$.

   Sketch the proof that if $G$ is a compact group then there exists a unique invariant measure on $G$ with $\mu(1) = 1$.

8. Determine the simple representations of $\mathrm{SO}(2)$.

   Determine the simple representations of $\mathrm{O}(2)$.

9. Determine the conjugacy classes in $\mathrm{SU}(2)$.

   Prove that $\mathrm{SU}(2)$ has one simple representation of each dimension $0, 1, 2, \ldots$, and determine the character of this representation.