Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. What is a group representation?
What is meant by saying that a representation is simple?
Determine from first principles all simple representations of $S(3)$.

2. What is meant by saying that the representation $\alpha$ is semisimple?
Prove that every finite-dimensional representation $\alpha$ of a finite group over $\mathbb{C}$ is semisimple.
Show that the natural representation of $S_n$ in $\mathbb{C}^n$ (by permutation of coordinates) splits into 2 simple parts, for any $n > 1$.

3. Determine the conjugacy classes in $S_4$, and draw up its character table.
Determine also the representation-ring for $S_4$, ie express the product $\alpha \beta$ of each pair of simple representations as a sum of simple representations.

More questions overleaf!
4. Prove that the number of simple representations of a finite group $G$ is equal to the number of conjugacy classes in $G$.

5. Show that if the finite group $G$ has simple representations $\sigma_1, \ldots, \sigma_s$ then
\[
\deg^2 \sigma_1 + \cdots + \deg^2 \sigma_s = |G|.
\]
Determine the degrees of the simple representations of $S_6$.

6. Show that the dodecahedron has 60 proper symmetries, and determine how these are divided into conjugacy classes.

7. Define a measure $\mu$ on a compact space $X$.
Sketch the proof that if $G$ is a compact group then there exists a unique invariant measure on $G$ with $\mu(1) = 1$.

8. Determine the simple representations of $SO(2)$.
Determine the simple representations of $O(2)$.

9. Determine the conjugacy classes in $SU(2)$.
Prove that $SU(2)$ has one simple representation of each dimension $0, 1, 2, \ldots$, and determine the character of this representation.