Course 3413 — Group Representations
Sample Paper III

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2 hour paper

Attempt 3 questions. (If you attempt more, only the best 3 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*.

   What is meant by saying that a representation α is *simple*? Determine all simple representations of S_3. from first principles.

   Determine the characters of S_4 induced by the simple characters of S_3. Hence or otherwise draw up the character table for S_4

   Answer:

   (a) *A representation α of a group G in a vector space V is a homomorphism*

   $$\alpha : G \to \mathrm{GL}(V).$$

   (b) *The representation α of G in V is said to be simple if no subspace $U \subset V$ is stable under G except for $U = 0, V$. (The subspace U is said to be stable under G if*

   $$g \in G, u \in U \implies gu \in U.)$$

(c) Writing s, t for the permutations $(123), (12)$ we have

$$S_3 = \langle s, t : s^3 = t^2 = 1, st = ts^2 \rangle.$$

Suppose α is a 1-dimensional representations of S_3. ie a homomorphism

$$\alpha : S_3 \to \mathbb{C}^*.$$

Let

$$\alpha(s) = \lambda,\ \alpha(t) = \mu.$$

Then

$$\lambda^3 = 1,\ \mu^2 = 1,\ \lambda\mu = \mu\lambda^2 \implies \lambda = 1, \mu = \pm 1.$$

Thus there are just 2 1-dimensional representations $1, \epsilon$ given by

$$s \mapsto 1, t \mapsto \pm 1.$$

Suppose α is a simple representation of degree $d > 1$ in the vector space V over $\mathbb{C}$. Let e be an eigenvector of s, say

$$se = \lambda e.$$

Then

$$s(te) = ts^2 e = \lambda^2 (te),$$

ie $f = te$ is a λ^2-eigenvector of s.
The vector subspace

$$U = \langle e, f \rangle$$

is stable under S_3 since

$$se = \lambda e,\ te = f\ sf = \lambda^2 f,\ tf = t^2 e = e.$$

Hence

$$V = U = \langle e, f \rangle.$$

Thus α is of degree 2.
Since $s^3 = 1$,

$$\lambda^3 = 1,$$

ie

$$\lambda \in \{1, \omega, \omega^2\},$$

where $\omega = e^{2\pi i/3}$.
If $\lambda = 1$ *then*

$$se = e, \; te = f \; sf = f, \; tf = e.$$

It follows that

$$s(e+f) = e+f, \; t(e+f) = e+f.$$

Thus the 1-dimensional subspace $\langle e+f \rangle$ *is stable under* S_3, *so* α *is not simple.*
Hence $\lambda = \omega$ *or* ω^2, *giving the representations*

$$s \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$s \mapsto \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These representations are equivalent, on swapping e, f.
Hence S_3 *has just 3 simple representations, of degrees 1,1,2.*

(d) *Recall that if* β *is a representation of* $H \subset G$, *then it induces a representation* α *of* G *with character*

$$\chi_\alpha([g]) = \frac{\#G}{\#H} \sum_{[h] \subset [g]} \frac{\#[h]}{\#[g]} \chi_\beta[h],$$

where the sum is over the classes $[h]$ *of* H *contained in the class* $[g]$ *of* G.
From above, the character table of S_3 *is*

	1^3	21	3
1	1	1	1
ϵ	1	-1	1
α	2	0	-1

S_4 *has 5 classes:* $1^4, 21^2, 2^2, 31, 4$. *We have*

$$\begin{aligned} 1^4 \cap S_3 &= 1^3, \\ 21^2 \cap S_3 &= 21, \\ 2^2 \cap S_3 &= \emptyset, \\ 31 \cap S_3 &= 3, \\ 4 \cap S_3 &= \emptyset. \end{aligned}$$

Also

$$[S_4 : S_3] = 24/6 = 4,\ [1^4 : 1^3] = 1/1 = 1,\ [21^2 : 21] = 6/3 = 2,\ [31 : 3] = 8/2 = 4.$$

Thus from our formula above the simple characters of S_3 induce the following characters of S_4:

	1^4	21^2	2^2	31	4
1^{S_4}	4	2	0	1	0
ϵ^{S_4}	4	-2	0	1	0
α^{S_4}	8	0	0	-1	0

We assume known the two representations of S_4 of degree 1:

	1^4	21^2	2^2	31	4
1	1	1	1	1	1
ϵ	1	-1	1	1	-1

We have

$$I(1^{S_4}, I^{S_4}) = \frac{1}{24}\left(4^2 + 6 \cdot 2^2 + 8 \cdot 1^2\right) = 2.$$

Thus 1^{S_4} splits into two parts, of which one is evidently the trivial representation, giving the simple representation

$$\phi = 1^{S_4} - 1$$

of S_4.
Similarly

$$\epsilon(1^{S_4}, \epsilon^{S_4}) = 2,$$

while

$$I(\epsilon^{S_4}, \epsilon) = \frac{1}{24}\left(4 \cdot 1 + 6 \cdot -2 \cdot -1 + 8 \cdot 1 \cdot 1\right) = 1.$$

Thus ϵ^{S_4} splits into two parts, of which one is ϵ, giving the simple representation

$$\psi = \epsilon^{S_4} - 1$$

Since S_4 has 5 classes, it has 5 simple representations, of which we have found 4:

	1^4	21^2	2^2	31	4
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
ϕ	3	1	-1	0	-1
ψ	3	-1	-1	0	1

(Evidently $\psi = \epsilon\phi$.)
Finally, we have

$$\alpha(1^{S_4}, \alpha^{S_4}) = \frac{1}{24}\left(8^2 + 6 \cdot +8 \cdot 1^2\right) = 3.$$

So α^{S_4} splits into three parts. We have

$$I(\alpha^{S_4}, 1) = \frac{1}{24}(8 - 8) = 0,$$

and similarly

$$I(\alpha^{S_4}, \epsilon) = \frac{1}{24}(8 - 8) = 0.$$

Thus the three parts of α^{S_4} must be ϕ, ψ and the last simple representation

$$\theta = \alpha^{S_4} - \phi - \psi,$$

enabling us to complete the table

	1^4	21^2	2^2	31	4
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
ϕ	3	1	-1	0	-1
ψ	3	-1	-1	0	1
θ	2	0	2	-1	0

2. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Determine the conjugacy classes in A_4 (formed by the even permutations in S_4), and draw up its character table.

   Determine also the representation-ring for this group, ie express the product $\alpha\beta$ of each pair of simple representation as a sum of simple representations.

   Answer:

   (a) *Let the simple representations of G be $\sigma_1, \ldots, \sigma_r$; and let $\chi_i(g)$ be the character of σ_i.*
   The simple characters $\chi_1, \ldots, \chi_r$ are linearly independent. For if say

$$\rho_1\chi_1(g) + \cdots + \rho_s\chi_s(g) = 0$$

it follows from the formula for the intertwining number that for any representation α

$$\rho_1 I(\alpha, \sigma_1) + \cdots + \rho_r I(\alpha, \sigma_r) = 0.$$

But on applying this with $\alpha = \sigma_i$ *we deduce that* $\rho_i = 0$ *for each* i.
The characters are class functions:

$$\chi(gxg^{-1}) = \chi(x).$$

The space of class functions has dimension s*, the number of classes in* G*. It follows that* $r \leq s$.
To prove that $r = s$*, it is sufficient to show that the characters span the space of class functions.*
Suppose $g \in G$ *has order* e*. Let* $[g]$ *denote the class of* g*, and let* $C = \langle g \rangle$ *be the cyclic group generated by* g.
The group C *has* e *1-dimensional representations* $\theta_1, \ldots, \theta_e$ *given by*

$$\theta_i : g \mapsto \omega^i,$$

where $\omega = e^{2\pi i/e}$.
Let

$$f(x) = \theta_0(x) + \omega^{-1}\theta_1(x) + \omega^{-2}\theta_2(x) + \cdots + \omega^{-e+1}\theta_{e-1}(x)$$

Then

$$f(g^j) = \begin{cases} e & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now let us "induce up" each of the characters θ_i *from* C *to* G*. We have*

$$\theta_i^G(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} \theta_i(y).$$

Let $F(x)$ *be the same linear combination of the induced characters that* $f(x)$ *was of the* θ_i*. Then*

$$F(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} f(y).$$

Since $f(y)$ *vanishes away from* g*, we deduce that* $F(x)$ *vanishes off the class* $[g]$*, and is non-zero on that class:*

$$F(x) \begin{cases} > 0 & \text{if } x \in [g], \\ = 0 & \text{if } x \notin [g]. \end{cases}$$

It follows that every class function on G can be expressed as a linear combination of characters, and therefore as a linear combination of simple characters. Hence the number of simple characters is at least as great as the number of classes.

We have shown therefore that the number of simple representations is equal to the number of classes.

(b)

Lemma 1. *Suppose $[g]$ is an even class in S_n. Then $[g]$ splits in A_n if and only if there is no odd permutation $t \in S_n$ which commutes with g; and if that is so then $[g]$ splits into two equal parts.*

Proof. By Lagrange's Theorem

$$\#[g] = \frac{\#S_n}{\#Z(g)},$$

where

$$Z(g) = \{x \in S_n : xg = gx\}.$$

Similarly, if $[g]'$ is the class of g in A_n, and

$$Z'(g) = \{x \in A_n : xg = gx\}$$

then

$$\#[g]' = \frac{\#A_n}{\#Z'(g)},$$

If no odd permutation commutes with g then

$$Z'(g) = Z(g),$$

and so

$$\#[g]' = \frac{1}{2}\#[g].$$

Since this is true for any A_n-class in $[g]$, it follows that $[g]$ splits into two equal classes.

Conversely, if there an odd permutation commutes with g then

$$\#Z'(g) < \#Z(g),$$

and so

$$\#[g]' > \frac{1}{2}\#[g].$$

Since this is true for any A_n-class in $[g]$ it follows that

$$[g]' = [g].$$

$\square$

There are 3 even classes in A_4: 1^4, 2^2, 31. Since the first two contain an odd number of elements they cannot split.
If x commutes with $(abc) \in 31$ then $x(d) = d$, and so

$$Z(abc) = \langle (abc) \rangle$$

contains no odd permutations.
Hence the class 31 splits. On considering the action of $V_4 = \{1, (ab)(cd), (ac)(bd), (ad)(bc)\}$ on (abc), it follows that the A_4-class of (abc) is

$$31' = \{(abc), (bad), (cad), (dcb),$$

while the inverses of these elements form the other class

$$31'' = \{(cba), (dab), (dac), (bcd)\}.$$

(c) Since

$$V_4 \lhd A_4,$$

with

$$A_4/V_4 = C_3.$$

It follows that the 3 representations of C_3 of degree 1 define 3 representations of A_4 of degree 1, which we may call $1, \omega, \omega^2$, with characters

	1^4	2^2	$31'$	$31''$
1	1	1	1	1
ω	1	1	ω	ω^2
ω^2	1	1	ω^2	ω

Since

$$1^2 + 1^2 + 1^2 + 3^2 = 12$$

there must be a 4th simple representation of degree 3 (and so the class 31 must split, as we have seen).
We know that the natural representation ρ of S_4 of degree 4 (defined by permutation of the coordinates) splits into two simple parts

$$\rho = 1 + \alpha,$$

with character

	1^4	21^2	2^2	31	4
ρ	4	3	0	1	0
α	3	2	-1	0	-1

The restriction $\theta = \alpha|A_4$ *has character*

	1^4	2^2	$31'$	$31''$
θ	3	-1	0	0

Since

$$I(\theta, \theta) = \frac{1}{12}\left(3^2 + 3 \cdot (-1)^2\right) = 1,$$

θ *is simple, allowing us to complete the character table for* A_4*:*

	1^4	2^2	$31'$	$31''$
1	1	1	1	1
ω	1	1	ω	ω^2
ω^2	1	1	ω^2	ω
θ	3	-1	0	0

(d) *Since the product of a simple representation with a representation of degree 1 is simple, and since* θ *is the only simple representation of degree 3,*

$$1\theta = \omega\theta = \omega^2\theta = \theta.$$

The products of the representations of degree 1 are trivial,

$$\omega \cdot \omega = \omega^2,$$

etc.

This leave the product θ^2 *of degree 9 to compute. Recall that*

$$I(\alpha\beta, \gamma) = I(\alpha, \beta^*\theta).$$

Since θ^* *is simple, and of degree 3,*

$$\theta^* = \theta,$$

and so

$$\begin{aligned} I(1, \theta^2) &= I(\theta^*, \theta) = I(\theta, \theta) = 1, \\ I(\omega, \theta^2) &= I(\omega\theta^*, \theta) = I(\theta, \theta) = 1, \\ I(\omega^2, \theta^2) &= I(\omega^2\theta^*, \theta) = I(\theta, \theta) = 1. \end{aligned}$$

It follows that

$$\theta^2 = 1 + \omega + \omega^2 + 2\theta.$$

In summary, the representation-ring is:

	1	ω	ω^2	θ
1	1	ω	ω^2	θ
ω	ω	ω^2	1	θ
ω^2	ω^2	1	ω	θ
θ	θ	θ	θ	$1 + \omega + \omega^2 + 2\theta$

3. Determine all groups of order 8 (up to isomorphism); and for each such group G determine as many simple representations (or characters) of G as you can.

   Answer:

   (a) *Suppose first that the group is abelian. From the Structure Theorem for Finite Abelian Groups, such a group is a direct product of cylic groups. In our case there are 3 such groups:*

   $$C_2 \times C_2 \times C_2,\ C_4 \times C_2,\ C_8.$$

   Now suppose the group G is non-abelian.

   We know that a group in which every element satisfies $g^2 = 1$ is necessarily abelian.

   Also, if there is an element of order 8 in the group then $G = C_8$ is abelian.

   It follows that there is at least one element $h \in G$ of order 4. Since the subgroup

   $$H = \langle h \rangle = C_4$$

   is of index 2 in G, it is normal in G:

   $$H \lhd G.$$

   Suppose there is an element $g \in G \setminus H$ of order 2. Then $g^{-1}hg \in H$ is of order 4; so

   $$g^{-1}hg = h \text{ or } h^{-1}.$$

   If $g^{-1}hg = h$ then G is abelian, and has already been determined. Thus

   $$g^{-1}hg = h^{-1} = h^3,$$

giving
$$G = \left\langle g, h : g^2 = h^4 = 1,\ hg = gh^3 \right\rangle,$$
which we regognise as D_4.

Finally, suppose all the elements of $G \setminus H$ are of order 4, Let g be one such element. Then
$$g^{-1}hg = h^{-1}.$$
as before. Also. $g^2 \in H$, since $[G : H] = 2$. Since g^2 is of order 2, it follows that
$$g^2 = h^2.$$
Thus
$$G = \left\langle g, h : h^4 = 1,\ g^2 = h^2,\ hg = gh^3 \right\rangle,$$
which we recognise as the quaternion group
$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

(b) *The simple representations of $H \times K$ are the representations $\alpha \times \beta$, where α, β are simple representations of H, K.*

Also
$$C_n = \langle g \rangle$$
has just n simple representations, all of degree 1, given by
$$g \mapsto \omega^r \quad (r = 0, 1, \dots, n-1),$$
where $\omega = e^{2\pi i/n}$.

It follows that a finite abelian group of order n has just n simple representations, all of degree 1.

In particular, each of the 3 abelian groups of order 8 has 8 simple representations of degree 1.

Turning to
$$D_4 = \left\langle s, t : s^4 = t^2 = 1, ts = s^3 t \right\rangle,$$
we see that this has 4 representations of degree 1, given by
$$s \mapsto \pm 1,\ t \mapsto \pm 1.$$

Recall that if the finite group G of order n has simple representations $\sigma_1, \dots, \sigma_s$ or degrees $d_1, \dots, d_s$ then
$$d_1^2 + \cdots + d_s^2 = n.$$

This implies that D_4 has one further simple representation, of degree 2. This is the natural representation of D_4, regarded as the symmetry group of a square, acting on $\mathbb{R}^2$, extended to C^2.

Thus D_4 has 5 simple representations, of degrees 1,1,1,1,2.

Similarly,

$$Q_8 = \langle s, t : s^4 = 1, t^2 = s^2, ts = s^3 t \rangle$$

has 4 representations of degree 1, given by

$$s \mapsto \pm 1, \; t \mapsto \pm 1.$$

As with D_4, it must have one further simple representation, of degree 2.

If we regard the complex numbers $\mathbb{C}$ as a sub-algebra of the quaternions $\mathbb{H}$ then $\mathbb{H}$ becomes a 2-dimensional vector space over $\mathbb{C}$, with basis $1, j$. This gives the required representation of $Q_8 \subset \mathbb{H}$, with $s = i, \; t = j$ acting by

$$s \mapsto \begin{pmatrix} i\&0 & \\ 0 & i \end{pmatrix}, \; t \mapsto \begin{pmatrix} 0\& -1 & \\ 1 & 0 \end{pmatrix}.$$

4. Determine the conjugacy classes in SU(2); and prove that this group has just one simple representation of each dimension.

   Define a covering homomorphism

   $$\Theta : \mathrm{SU}(2) \to \mathrm{SO}(3);$$

   and hence or otherwise show that SO(3) has one simple representation of each odd dimension $1, 3, 5, \ldots$.

   Answer:

   (a) *Suppose $U \in \mathrm{SU}(2)$. The eigenvalues λ, μ of U satisfy*

   $$|\lambda| = |\mu| = 1, \; \lambda\mu = 1.$$

   Hence we can write the eigenvalues as

   $$e^{\pm i\theta},$$

   where we may take $-\pi < \theta \leq \pi$.

   We can diagonalize U, ie find $V \in \mathrm{SU}(2)$ such that

   $$V^{-1}UV = U(\theta) = \begin{pmatrix} e^{i\theta}0 & \\ 0 & e^{-i\theta} \end{pmatrix}.$$

(For if e is a λ-eigenvector with $e^*e = 1$, then we can extend to an orthonormal basis e, f, and setting $V = (e, f)$ we have

$$V^{-1}UV = \begin{pmatrix} \lambda & b \\ c & d \end{pmatrix}.$$

But since this matrix is unitary,

$$\lambda^*\lambda + b^*b = 1, \ \lambda^*\lambda + c^*c = 1$$

and so $b = c = 0$.)
It is evident that

$$U(-\theta) \sim U(\theta),$$

on swapping the basis elements.
On the other hand, since

$$\operatorname{tr} U(\theta) = 2\cos\theta$$

it follows that the matrices $U(\theta)$ $0 \le \theta < \pi$ are not similar.
Hence there is a conjugacy class $C(\theta)$ corresponding to each $\theta \in [0, \pi]$, consisting of all U with

$$\operatorname{tr} U = 2\cos\theta.$$

(b) *Suppose $n \in \mathbb{N}$, Let $V(n)$ denote the space of homogeneous polynomials $P(z, w)$ of degree n in z, w. Thus $V(n)$ is a vector space over $\mathbb{C}$ of dimension $n + 1$, with basis $z^n, z^{n-1}w, \ldots, w^n$.*
Suppose $U \in \mathrm{SU}(2)$. Then U acts on z, w by

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

This action in turn defines an action of $\mathrm{SU}(2)$ on $V(n)$:

$$P(z, w) \mapsto P(z', w').$$

We claim that the corresponding representation of $\mathrm{SU}(2)$ — which we denote by $D_{n/2}$ — is simple, and that these are the only simple (finite-dimensional) representations of $\mathrm{SU}(2)$ over $\mathbb{C}$.
To prove this, let

$$\mathrm{U}(1) \subset \mathrm{SU}(2)$$

be the subgroup formed by the diagonal matrices $U(\theta)$. The action of $\mathrm{SU}(2)$ *on z, w restricts to the action*

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

of $\mathrm{U}(1)$. *Thus in the action of* $\mathrm{U}(1)$ *on $V(n)$,*

$$z^{n-r} w^r \mapsto e^{(n-2r)i\theta} z^{n-r} w^r,$$

It follows that the restriction of $D_{m/1}$ to $U(1)$ is the representation

$$D_{n/2} | \mathrm{U}(1) = E(n) + E(n-2) + \cdots + E(-n)$$

where $E(n)$ is the representation

$$e^{i\theta} \mapsto e^{ni\theta}$$

of $\mathrm{U}(1)$.
In particular, the character of $D_{n/2}$ is given by

$$\chi_{n/2}(U) = e^{ni\theta} + e^{(n-2} i\theta + \cdots + e^{-ni\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.
Now suppose $D_{n/2}$ is not *simple, say*

$$D_{n/2} = \alpha + \beta.$$

(We know that $D_{n/2}$ is semisimple, since $\mathrm{SU}(2)$ *is compact.) Let a corresponding split of the representation space be*

$$V(n) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2} |$ $\mathrm{U}(1)$ *are distinct, the expression of $V(n)$ as a direct sum of* $\mathrm{U}(1)$*-spaces,*

$$V(n) = \langle z^n \rangle \oplus \langle z^{n-1} w \rangle \oplus \cdots \oplus \langle w^n \rangle$$

is unique. It follows that W_1 must be the direct sum of some of these spaces, and W_2 the direct sum of the others. In particular $z^n \in W_1$ or $z^n \in W_2$, say $z^n \in W_1$. Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Then
$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z + w \\ -z + w \end{pmatrix}$$
under U. Hence
$$z^m \mapsto 2^{-m/2}(z + w)^m.$$
Since this contains non-zero components in each subspace $\langle z^{m-r} w^r \rangle$, it follows that
$$W_1 = V(n),$$
ie the representation $D_{m/2}$ of SU(2) *in $V(m)$ is simple.*

To see that every simple (finite-dimensional) representation of SU(2) *is of this form, suppose α is such a representation. Consider its restriction to* U(1)*. Suppose*

$$\alpha |\, \mathrm{U}(1) = e_r E(r) + e_{r-1} E(r-1) + \cdots + e_{-r} E(-r) \quad (e_r, e_{r-1}, \ldots, e_{-r} \in \mathbb{N}).$$

Then α has character

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1} e^{(r-1)i\theta} + \cdots + e_{-r} e^{-ri\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.

Since $U(-\theta) \sim U(\theta)$ it follows that

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r (e^{ri\theta} + e^{-ri\theta}) + e_{r-1} (e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots .$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ with integer (possibly negative) coefficients:

$$\chi(\theta) = a_0 \chi_0(\theta) + a_{1/2} \chi_{1/2}(\theta) + \cdots + a_s \chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$). Since α is simple,

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j is ± 1 and the rest are 0, ie

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j. But

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$

(c) *Let V be the space of hermitian 2×2-matrices with trace 0. (Thus V consists of the matrices of form*

$$X = \begin{pmatrix} x & y - iz \\ y + iz & -x \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$.)
Then V is a real vector space of dimension 3, with basis

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; G = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Suppose $U \in \mathrm{SU}(2),\ X \in V$. Let

$$Y = U^{-1}XU = U^*XU.$$

Then

$$Y^* = U^*X^*U = U^*XU = Y,$$

ie Y is hermitian; and

$$\operatorname{tr} Y = \operatorname{tr} X = 0.$$

Thus $Y \in V$, and we have defined an action of $\mathrm{SU}(2)$ on V, giving a homomorphism

$$\Theta : SU(2) \to GL(3, \mathbb{R}).$$

We can define a positive-definite inner product on V by

$$\langle X, Y \rangle = \operatorname{tr}(X^* Y);$$

and this inner-product is preserved by the action, ie

$$\operatorname{tr}((U^* X U)^* U^* Y U) = \operatorname{tr}(X^* Y).$$

Thus

$$\operatorname{im} \Theta \subset \mathrm{O}(3).$$

In fact, since $SU(2)$ is connected (it is homeomorphic to the 3-sphere S_3) it follows that

$$\operatorname{im} \Theta \subset \mathrm{SO}(3).$$

Thus we have defined a homomorphism

$$\Theta : SU(2) \to \mathrm{SO}(3).$$

We have

$$\ker \Theta = \{\pm I\}.$$

For if $U \in \ker \Theta$ then

$$UX = XU$$

for all $X \in V$.
Since any hermitian matrix can be written as $\lambda I + X$, with $X \in V$, it follows that

$$UX = XU$$

for all hermitian matrices.
Since ever skew-hermitian matrix is of the form iX, with X hermitian, it follows that

$$UX = XU$$

for all skew-hermitian matrices; and since every matrix is the sum of a hermitian and a skew-hermitian matrix, it follows that

$$UX = XU$$

for all 2×2-matrices over $\mathbb{C}$. Hence

$$U = \pm I.$$

It remains to show that Θ *is surjective, ie*

$$\operatorname{im}\Theta = \mathrm{SO}(3).$$

It is easy to see that

$$\Theta U(\theta) = R(2\theta, Ox)$$

(the rotation in V *through angle* 2θ *about the axis* Ox*).*

Also $\Theta(F)$, $\Theta(G)$ *(where* F, G *are the matrices defined above) are half-turns about* Oy, Oz *respectively.*

Since these rotations generate $\mathrm{SO}(3)$ *we have shown that* Θ *is surjective.*

(d) If we have a surjective homomorphism

$$\theta : G \to H,$$

then each representation of H *defines a representation of* G *(in the same vector space); and a representation of* G *arises in this way precisely if it is trivial on* $\ker\theta$*. Also the representation of* G *is simple if and only if the representation of* H *is simple.*

It follows that the simple representations of $\mathrm{SO}(3)$ *correspond to the simple representations of* $\mathrm{SU}(2)$ *which are trivial on* $\{\pm I\}$*.*

But it is easy to see that $-I$ *acts on* $V(n)$ *by*

$$P(z, w) \mapsto (-1)^n P(z, w).$$

Thus the action is trivial if and only if n *is even, ie if and only if the degree* $n + 1$ *of the representation is odd.*

In other words, the simple representations of $\mathrm{SO}(3)$ *correspond to the representations* D_j *of* $\mathrm{SU}(2)$ *with* $j \in \mathbb{N}$*.*