Course 3413 — Group Representations
Sample Paper II

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Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. What is a *group representation*?

   What is meant by saying that a representation is *simple*?

   What is meant by saying that a representation is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

2. Show that all simple representations of an abelian group are of degree 1.

   Determine from first principles all simple representations of $D(6)$.

3. Determine all groups of order 30.

4. Prove that the number of simple representations of a finite group G is equal to the number of conjugacy classes in G.

More questions overleaf!

Show that if the finite group G has simple representations $\sigma_1, \dots, \sigma_s$ then

$$\deg^2 \sigma_1 + \cdots + \deg^2 \sigma_s = |G|.$$

Determine the degrees of the simple representations of S_6.

5. Determine the conjugacy classes in A_5, and draw up the character table of this group.

6. If α is a representation of the finite group G and β is a representation of the finite group H, define the representation $\alpha \times \beta$ of the product-group $G \times H$.

   Show that if α and β are simple then so is $\alpha \times \beta$, and show that every simple representation of $G \times H$ is of this form.

   Show that the symmetry group G of a cube is isomorphic to $C_2 \times S_4$.

   Into how many simple parts does the permutation representation of G defined by its action on the vertices of the cube divide?

7. Show that every representation of a compact group is semisimple.

   Determine the simple representations of $\mathrm{U}(1)$.

   Verify that the simple characters of $\mathrm{U}(1)$ are orthogonal.

8. Show that $\mathrm{SU}(2)$ has just one simple representation of each degree $0, 1, 2, \dots$.

   Determine the simple representations of $\mathrm{U}(2)$.

9. Show that $\mathrm{SU}(2)$ is isomorphic to $\mathrm{Sp}(1)$ (the group of unit quaternions).

   Define a 2-fold covering $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, and so determine the simple representations of $\mathrm{SO}(3)$.