Course 424

Group Representations

Dr Timothy Murphy

Sample Paper

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. What is meant by saying that a group representation α is (a) *simple*, (b) *semisimple*?

   Prove that every representation of a finite group is semisimple.

   Give an example of a representation of an infinite group that is not semisimple.

   Answer:

   (a) The representation α of G in V is said to be simple if no subspace $U \subset V$ is stable under G except for $U = 0, V$. (The subspace U is said to be stable under G if

   $$g \in G, u \in U \implies gu \in U.)$$

   (b) The representation α of G in V is said to be semisimple if it can be expressed as a sum of simple representations:

   $$\alpha = \sigma_1 + \cdots + \sigma_m.$$

This is equivalent to the condition that each stable subspace $U \subset V$ has a stable complement W:

$$V = U \oplus W.$$

(c) *Suppose α is a representation of the finite group G in the vector space V. Let*

$$P(u, v)$$

be a positive-definite hermitian form on V. Define the hermitian form Q on V by

$$Q(u, v) = \frac{1}{\|G\|} \sum_{g \in G} H(gu, gv).$$

Then Q is positive-definite (as a sum of positive-definite forms). Moreover Q is invariant under G, ie

$$Q(gu, gv) = Q(u, v)$$

for all $g \in G, u, v \in V$. For

$$\begin{aligned} Q(hu, hv) &= \frac{1}{\|G\|} \sum_{g \in G} H(ghu, ghv) \\ &= \frac{1}{|G|} \sum_{g \in G} H(gu, gv) \\ &= Q(u, v), \end{aligned}$$

since gh runs over G as g does.

Now suppose U is a stable subspace of V. Then

$$U^{\perp} = \{v \in V : Q(u, v) = 0 \ \forall u \in U\}$$

is a stable complement to U.

Thus every stable subspace has a stable complement, ie the representation is semisimple.

(d) *The representation α of $\mathbb{Z}$ of degree 2 over $\mathbb{C}$ given by*

$$n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

is not semisimple.

For the representation is not simple, since it leaves stable the 1-dimensional subspace $\langle e \rangle$, *where*

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If α *were semisimple, say* $\alpha = \beta + \gamma$, *where* β, γ *are of degree 1, then* $\alpha(n)$ *would be diagonalisable for all* n.

Since $\alpha(n)$ *has eigenvalues* $1, 1$, *this implies that*

$$\alpha(n) = I$$

for all n, *which is not the case.*

2. Draw up the character table for S_4.

   Determine also the representation-ring for S_4, ie express the product $\alpha\beta$ of each pair of simple representations as a sum of simple representations.

   Draw up the character table for the subgroup A_4 of even permutations.

   Answer:

   (a) *S_4 has 5 classes, corresponding to the types* $1^4, 1^2 2, 13, 2^2, 4$. *Thus S_4 has 5 simple representations.*

   Each symmetric group S_n *(for* $n \geq 2$*) has just 2 1-dimensional representations, the trivial representation* 1 *and the parity representation* ϵ.

   Let $S_4 = \mathrm{Perm}(()X)$, *where* $X = \{a, b, c, d\}$. *The action of* S_4 *on* X *defines a 4-dimensional representation* ρ *of* S_4, *with character*

$$\chi(g) = |\{x \in X : gx = x\}|$$

   In other words $\chi(g)$ *is just the number of 1-cycles in* g.

   So now we can start our character table (where the second line gives the number of elements in the class):

	1^4	$1^2 2$	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	−1	1	1	−1
ρ	4	2	1	0	0

Now

$$I(\rho, \rho) = \frac{1}{24}(1 \cdot 16 + 6 \cdot 4 + 8 \cdot 1) = 2.$$

It follows that ρ has just 2 simple parts. Since

$$I(1, \rho) = \frac{1}{24}(1 \cdot 4 + 6 \cdot 2 + 8 \cdot 1) = 1,$$

It follows that

$$\rho = 1 + \alpha,$$

where α is a simple 3-dimensional representation, with character given by

$$\chi(g) = \chi_\rho(g) - 1.$$

The representation $\epsilon\alpha$ is also simple, and is not equal to α since it has a different character. So now we have 4 simple characters of S_4, as follows:

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1

To find the 5th simple representation, we can consider α^2. This has character

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
α^2	9	1	0	1	1

We have

$$\begin{aligned}
I(1, \alpha^2) &= \frac{1}{24}(9 + 6 + 3 + 6) = 1, \\
I(\epsilon, \alpha^2) &= \frac{1}{24}(9 - 6 + 3 - 6) = 0, \\
I(\alpha, \alpha^2) &= \frac{1}{24}(27 + 6 - 3 - 6) = 1, \\
I(\epsilon\alpha, \alpha^2) &= \frac{1}{24}(27 - 6 - 3 + 6) = 1. I(\alpha^2, \alpha^2) \quad = \frac{1}{24}(81 + 6 + 3 + 6) = 4,
\end{aligned}$$

It follows that α^2 has 4 simple parts, so that

$$\alpha^2 = 1 + \alpha + \epsilon\alpha + \beta,$$

where β is the 5th simple representation, with character given by

$$\chi_\beta(g) = \chi_\alpha(g)^2 - 1 - \chi_\alpha(g) - \epsilon(g)\chi_\alpha(g).$$

This allows us to complete the character table:

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1
β	2	0	-1	2	0

(b) *We already know how to express α^2 in terms of the 5 simple representations. Evidently $\epsilon\beta = \beta$ since there is only 1 simple representation of dimension 2. The character of $\alpha\beta$ is given by*

	1^4	$1^2 2$	13	2^2	4
$\alpha\beta$	6	0	0	-2	0

We have

$$I(\alpha\beta, \alpha\beta) = \frac{1}{24}(36 + 12) = 2.$$

Thus $\alpha\beta$ has just 2 simple parts. These must be α and $\epsilon\alpha$ to give dimension 6:

$$\alpha\beta = \alpha + \epsilon\alpha.$$

Also we have

$$I(\beta^2, \beta^2) = \frac{1}{24}(16 + 8 + 48) = 3.$$

Thus β has 3 simple parts. So by dimension, we must have

$$\beta^2 = 1 + \epsilon + \beta.$$

Now we can give the multiplication table for the representation-ring:

	1	ϵ	β	α	$\epsilon\alpha$
1	1	ϵ	β	α	$\epsilon\alpha$
ϵ	ϵ	1	β	$\epsilon\alpha$	α
β	β	β	$1 + \epsilon + \beta$	$\alpha + \epsilon\alpha$	$\alpha + \epsilon\alpha$
α	α	$\epsilon\alpha$	$\alpha + \epsilon\alpha$	$1 + \beta + \alpha + \epsilon\alpha$	$\epsilon + \beta + \alpha + \epsilon\alpha$
$\epsilon\alpha$	$\epsilon\alpha$	α	$\alpha + \epsilon\alpha$	$\epsilon + \beta + \alpha + \epsilon\alpha$	$1 + \beta + \alpha + \epsilon\alpha$

(c) *Recall that an even class $\bar{g} \subset S_n$ splits in A_n if and only if no odd element $x \in S_n$ commutes with g, in which case $\bar{g}$ splits into two classes of equal size.*

There are 3 even classes in S_n: $1^4, 2^2 and 31$, containing $1, 3, 8$ elements, respectively. The first two cannot split, since they contain an odd number of elements. The third class does split; for suppose x commutes with $g = (abc)$. Then

$$xgx^{-1} = (x(a), x(b), x(c)) = (a, b, c).$$

It follows from this that

$$x \in \{1, g, g^2\}.$$

In particular, x is even.

Thus the class 31 splits into two classes $31'$ and $31''$, each containing 4 elements.

3. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then

$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the degrees of the simple representations of S_6.

   Answer:

(a)

(b)

(c)

(d) *S_6 has 11 classes:*

$$1^6,\ 21^4,\ 2^21^2,\ 2^3,\ 31^3,\ 321,\ 3^2,\ 3^2,\ 41^2,\ 51,\ 6.$$

Hence it has 11 simple representations over $\mathbb{C}$.

It has 2 representations of degree 1: 1 and the parity representation ϵ.

The natural representation ρ_1 of degree 6 (by permutation of coordinates) splits into two simple parts:

$$\rho_1 = 1 + \sigma_1,$$

where σ_1 is of degree 5.
If α is a simple representation of odd degree, then

$$\epsilon\alpha \neq \alpha.$$

For a transposition t has eigenvalues ± 1, since $t^2 = 1$. Hence

$$\chi_\alpha(t) \neq 0.$$

But

$$\chi_{\epsilon\alpha}(t) = \chi_\epsilon(t)\chi_\alpha(t) = -\chi_\alpha(t).$$

Thus the simple representations of odd degree d divide into pairs α, $\epsilon\alpha$. So there are an even number of representations of degree d.
In particular there are at least 2 simple representations of degree 5: σ and $\epsilon\sigma$.
We are going to draw up a partial character table for S_6, adding rows as we gather more material.

	1^6	21^4	2^21^2	2^3	31^3	321	3^2	42	41^2	51	6
#	1	15	45	15	40	120	40	90	90	144	120
ρ_1	6	4	2	0	3	1	0	2	0	1	0
σ_1	5	3	1	-1	2	0	-1	1	-1	0	-1
ρ_2	15	7	3	3	3	1	0	1	1	0	0
τ	14	6	2	2	2	0	-1	0	0	-1	-1
σ_2	9	3	1	3	0	0	0	-1	1	-1	0
ρ_3	20	8	4	0	2	2	2	0	0	0	0
θ	19	7	3	-1	1	1	1	-1	-1	-1	-1
σ_3	5	1	1	-3	-1	1	2	-1	-1	0	0
σ_1^2	25	9	1	1	4	0	1	1	1	0	1
ϕ	24	8	0	0	3	-1	0	0	0	-1	0

Now consider the permutation representation ρ_2 arising from the action of S_6 on the 15 pairs of elements. Evidently

$$I(\rho_2, 1) > 0,$$

since all the terms in the sum for this are ≥ 0. Let $\tau = \rho_2 - 1$. Then

$$I(\tau, \tau) = \frac{1}{720}(196 + 540 + 180 + 60 + 160 + 40 + 144 + 120) = 2,$$

while

$$I(\tau, \sigma_1) = \frac{1}{720}(70 + 270 + 90 - 30 + 160 + 40 + 120) = 1.$$

Thus

$$\sigma_2 = \tau - \sigma_1$$

is simple.
So far we have 6 simple representations:

$$1, \epsilon, \sigma_1, \epsilon\sigma_1, \sigma_2, \epsilon\sigma_2,$$

of degrees 1,1,5,5,9,9.
Next consider the permutation representation ρ_3 arising from the action of S_6 on the 20 subsets of 3 elements. Evidently

$$I(\rho_3, 1) > 0,$$

since all the terms in the sum for this are ≥ 0.
[Although not needed here, it is worth recalling that if ρ is a permutation representation arising from the action of G on the set X then $I(\rho, 1)$ is equal to the number of orbits of the action.]
Let $\theta = \rho_3 - 1$. Then

$$I(\theta, \theta) = \frac{1}{720}(361+735+405+15+40+120+40+90+90+144+120) = 3.$$

Thus θ has 3 simple parts.
Now

$$I(\theta, \sigma_1) = \frac{1}{720}(95+315+135+15+80-40-90+90+120) = 1,$$

while

$$I(\theta, \sigma_2) = \frac{1}{720}(171 + 315 + 135 - 45 + 90 - 90 + 144) = 1.$$

It follows that

$$\sigma_3 = \theta - \sigma_1 - \sigma_2$$

is simple.
Now we have 8 simple representations:

$$1, \epsilon, \sigma_1, \epsilon\sigma_1, \sigma_3, \epsilon\sigma_3, \sigma_2, \epsilon\sigma_2,$$

of degrees 1,1,5,5,5,5,9,9.

We have 3 remaining simple representations. Suppose they are of degrees a, b, c. Then

$$720 = 2 \cdot 1^2 + 4 \cdot 5^2 + 2 \cdot 9^2 + a^2 + b^2 + c^2$$

ie

$$a^2 + b^2 + c^2 = 456.$$

Now

$$456 \equiv 0 \bmod 8.$$

If n is odd then $n^2 \equiv 1 \bmod 8$. It follows that a, b, c are all even, say

$$a = 2d,\ b = 2e,\ c = 2f,$$

with

$$d^2 + e^2 + f^2 = 114.$$

Since

$$114 \equiv 2 \bmod 8,$$

it follows that two of d, e, f are odd and one is divisible by 4. Let us suppose these are d, e, f in that order. Then

$$f \in \{4, 8\}.$$

If $f = 4$ then

$$d^2 + e^2 = 98 \implies d = e = 7,$$

while if $f = 8$ then

$$d^2 + e^2 = 50 \implies d = e = 5.$$

So the three remaining simple representations have degrees

$$8, 14, 14 \text{ or } 10, 10, 16.$$

Let

$$\phi = \sigma_1^2 - 1.$$

Then

$$I(\phi, \phi) = \frac{1}{720}(576 + 960 + 360 + 120 + 144) = 3.$$

Also

$$I(\phi, \sigma_1) = \frac{1}{720}(120 + 360 + 240) = 1,$$

while

$$I(\phi, \sigma_2) = \frac{1}{720}(216 + 360 + 144) = 1.$$

Thus

$$\sigma_4 = \phi - \sigma_1 - \sigma_2$$

is a simple representation of degree 10.

We conclude that the 11 simple representations have degrees

$$1, 1, 5, 5, 5, 5, 9, 9, 10, 10, 16.$$

4. Determine the simple representations of $\mathrm{SO}(2)$.

   Suppose H is a subgroup of the compact group G of finite index. Explain how a representation β of H induces a representation β^G of G.

   Determine the simple representations of $\mathrm{O}(2)$.

   Answer:

   (a) *Let*

   $$R(\theta) \in \mathrm{SO}(2)$$

   denote rotation through angle θ. Then the map

   $$R(\theta) \mapsto e^{i\theta} : \mathrm{SO}(2) \to \mathrm{U}(1)$$

   is an isomorphism, allowing us to identify $\mathrm{SO}(2)$ with $\mathrm{U}(1)$.

   This group is abelian; so every simple representation α (over $\mathbb{C}$) is of degree 1; and since the group is compact

   $$\operatorname{im} \alpha \subset \mathrm{U}(1).$$

   ie α is a homomorphism

   $$\mathrm{U}(1) \to \mathrm{U}(1).$$

   For each $n \in \mathbb{Z}$ the map

   $$E(n) : z \to z^n$$

   defines such a homomorphism. We claim that every representation of $\mathrm{U}(1)$ is of this form.

For suppose

$$\alpha : U(1) \to U(1)$$

is a representation of $\mathrm{U}(1)$ *distinct from all the* $E(n)$.
Then

$$I(E_n, \alpha) = 0$$

for all n, *ie*

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \alpha(e^{i\theta}) e^{-in\theta}\, d\theta = 0.$$

In other words, all the Fourier coefficients of $\alpha(e^{i\theta})$ vanish.

But this implies (from Fourier theory) that the function itself must vanish, which is impossible since $\alpha(1) = 1$.

(b) *Suppose* β *is a representation of* H *in the vector space* U.
Express G *as a union of left* H*-cosets:*

$$G = g_1 H \cup \cdots \cup g_r H$$

Set

$$V = g_1 U \oplus \cdots \oplus g_r U,$$

ie V *is the direct sum of* r *copies of* U, *labelled by* $g_1, \ldots, g_r$.
We define the action of $g \in G$ *on* V *as follows. Suppose* $1 \le i \le r$.
Then

$$g g_i = g_j h$$

for some $j \in [1, r]$, $h \in H$.
We set

$$g(g_i u) = g_j(hu).$$

That defines the action of g *on the summand* $g_i U$*; and this is extended to* V *by linearity.*

It is readily verified that this defines a representation of G *in* V, *and that the choice of different representatives* $g_1, \ldots, g_r$ *of the cosets would lead to an equivalent representation.*

(c) *Since* $\mathrm{SO}(2)$ *is a subgroup of index 2 in* $\mathrm{O}(2)$, *the representation* $E(n)$ *of* $\mathrm{SO}(2) = U(1)$ *induces a representation*

$$\alpha_n = E(n)^{\mathrm{O}(2)}$$

of $\mathrm{O}(2)$ *of degree 2.*

Any element of $\mathrm{O}(2) \setminus \mathrm{SO}(2)$ *is a reflection* $T(l)$ *in some line* l *through the origin. These reflections are all conjugate, since*

$$R(\theta)T(l)R(-\theta) = T(l'),$$

where $l' = R(\theta)l$.
Also

$$T(l)R(\theta)T(l) = R(-\theta);$$

so the $\mathrm{O}(2)$*-conjugacy classes consist of pairs* $\{R(\pm\theta)\}$*, together with the set of all reflections.*
Explicitly, on taking e, Te *as basis for the induced representation (where* T *is any reflection) we see that* α_n *is given by*

$$R(\theta) \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad T(l) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0. \end{pmatrix}.$$

If $n \neq 0$ *this representation is simple. For*

$$\alpha_n | \mathrm{SO}(2) = E(n) + E(-n).$$

It follows that the only proper subspaces stable under $\mathrm{SO}(2)$ *are* $\langle e \rangle$*,* $\langle Te \rangle$*, and these are not stable under* T.
If $n = 0$ *the representation splits into two parts:*

$$\alpha_0 = 1 + \epsilon,$$

where

$$\epsilon(R(\theta)) = 1, \ \epsilon(T(l) = -1,$$

ie $\epsilon(S) = \pm 1$ *according as* S *is proper or improper.*
We claim that the simple representations of $\mathrm{O}(2)$ *are precisely these representations* α_n *for* $n \neq 0$*, together with the representations* $1, \epsilon$ *of degree 1.*
For suppose α *is a simple representation of* $\mathrm{O}(2)$ *in the vector space* V*. Then*

$$\alpha | \mathrm{SO}(2) = E(n_1) + \cdots + E(n_r),$$

ie V *is the direct sum of 1-dimensional subspaces stable under* $\mathrm{SO}(2)$.
Let $U = \langle e \rangle$ *be one such subspace. Then* U *carries some representation* $E(n)$*, ie*

$$R(\theta)e = e^{in\theta}e$$

for all θ.

Take any reflection T. *Then the subspace* $\langle e, Te \rangle$ *is stable under the full group* $O(2)$. *Since* α *is simple,*

$$V = \langle e, Te \rangle,$$

If $n \neq 0$ *then we see explicitly that*

$$\alpha = \alpha_n.$$

If $n = 0$ *then* $SO(2)$ *acts trivially on* U. *If* $Te = e$ *then* U *is 1-dimensional, and* $\alpha = 1$. *If not, then the 1-dimensional subspace* $\langle e - Te \rangle$ *carries the representation* ϵ, *and so* $\alpha = \epsilon$.

We conclude that these are the only simple representations of $O(2)$.

5. Prove that $SU(2)$ has one simple representation of each dimension $0, 1, 2, \ldots$.

   Show that there exists a double covering $\Theta : SU(2) \to SO(3)$.

   Hence or otherwise determine the simple representations of $SO(3)$.

   Determine the representation-ring of $SO(3)$, ie express the product of each pair of simple representations as a sum of simple representations.

   Determine the simple representations of $O(3)$.

   Answer:

   (a) *Suppose* $m \in \mathbb{N}$, *Let* $V(m)$ *denote the space of homogeneous polynomials* $P(z, w)$ *in* z, w. *Thus* $V(m)$ *is a vector space over* $\mathbb{C}$ *of dimension* $m + 1$, *with basis* $z^m, z^{m-1}w, \ldots, w^m$.

   Suppose $U \in SU(2)$. *Then* U *acts on* z, w *by*

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

   This action in turn defines an action of $SU(2)$ *on* $V(m)$:

$$P(z, w) \mapsto P(z', w').$$

   We claim that the corresponding representation of $SU(2)$ — *which we denote by* $D_{m/2}$ — *is simple, and that these are the only simple (finite-dimensional) representations of* $SU(2)$ *over* $\mathbb{C}$.

To prove this, let

$$\mathrm{U}(1) \subset \mathrm{SU}(2)$$

be the subgroup formed by the diagonal matrices $U(\theta)$. *The action of* $\mathrm{SU}(2)$ *on* z, w *restricts to the action*

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

of $\mathrm{U}(1)$. *Thus in the action of* $\mathrm{U}(1)$ *on* $V(m)$,

$$z^{m-r} w^r \mapsto e^{(m-2r)i\theta} z^{m-r} w^r,$$

It follows that the restriction of $D_{m/1}$ *to* $U(1)$ *is the representation*

$$D_{m/2} | \,\mathrm{U}(1) = E(m) + E(m-2) + \cdots + E(-m)$$

where $E(m)$ *is the representation*

$$e^{i\theta} \mapsto e^{mi\theta}$$

of $\mathrm{U}(1)$.
In particular, the character of $D_{m/2}$ *is given by*

$$\chi_{m/2}(U) = e^{mi\theta} + e^{(m-2)i\theta} + \cdots + e^{-mi\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$.
Now suppose $D_{m/2}$ *is* not *simple, say*

$$D_{m/2} = \alpha + \beta.$$

(We know that $D_{m/2}$ *is semisimple, since* $\mathrm{SU}(2)$ *is compact.) Let a corresponding split of the representation space be*

$$V(m) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2} | \,\mathrm{U}(1)$ *are distinct, the expression of* $V(m)$ *as a direct sum of* $\mathrm{U}(1)$*-spaces,*

$$V(m) = \langle z^m \rangle \oplus \langle z^{m-1} w \rangle \oplus \cdots \oplus \langle w^m \rangle$$

is unique.
It follows that W_1 *must be the direct sum of some of these spaces, and* W_2 *the direct sum of the others. In particular* $z^m \in W_1$ *or* $z^m \in W_2$, *say* $z^m \in W_1$.

Let

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Then

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} z+w \\ -z+w \end{pmatrix}$$

under U. *Hence*

$$z^m \mapsto 2^{-m/2}(z+w)^m.$$

Since this contains non-zero components in each subspace $\langle z^{m-r}w^r \rangle$, *it follows that*

$$W_1 = V(m),$$

ie the representation $D_{m/2}$ *of* $\mathrm{SU}(2)$ *in* $V(m)$ *is simple.*
To see that every simple (finite-dimensional) representation of $\mathrm{SU}(2)$ *is of this form, suppose* α *is such a representation. Consider its restriction to* $\mathrm{U}(1)$. *Suppose*

$$\alpha|\,\mathrm{U}(1) = e_r E(r) + e_{r-1}E(r-1) + \cdots + e_{-r}E(-r) \quad (e_i \in \mathbb{N}).$$

Then α *has character*

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1}e^{(r-1)i\theta} + \cdots + e_{-r}e^{-ri\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$.
Since $U(-\theta) \sim U(\theta)$ *it follows that*

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r(e^{ri\theta} + e^{-ri\theta}) + e_{r-1}(e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots .$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ *with integer (possibly negative) coefficients:*

$$\chi(\theta) = a_0\chi_0(\theta) + a_{1/2}\chi_{1/2}(\theta) + \cdots + a_s\chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha,\alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$). Since α is simple,

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j is ± 1 and the rest are 0, ie

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j. But

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$

(b) *We can identify $\mathrm{SU}(2)$ with the group*

$$\mathrm{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\}.$$

[If we regard $\mathbb{H}$ as a 2-dimensional vector space over $\mathbb{C}$ with basis $1, j$:

$$(z, w) \mapsto z + wj,$$

then multiplication on the right by a quaternion defines a C-linear map, ie an element of $\mathrm{GL}(2, \mathbb{C})$.
Suppose $q = a + bj \in \mathrm{Sp}(1)$. Then

$$q^{-1} = q^* = \bar{a} - bj;$$

and multiplication on the right by q^{-1} gives the map

$$z + wj \mapsto (\bar{a}z + \bar{b}w) + (-bz + aw)j,$$

ie

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

Since

$$|q|^2 = |a|^2 + |b|^2,$$

this establishes an isomorphism

$$q \mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix} : \mathrm{Sp}(1) \to \mathrm{SU}(2).]$$

Now let V denote the 3-dimensional real vector space of purely imaginary quaternions

$$v = xi + yj + zk.$$

Evidently

$$q \in V \iff q^* = -q.$$

It follows that if $q \in \mathrm{Sp}(1),\ v \in V$ then

$$(qvq^*)^* = qv^*q^* = -qvq^ast.$$

Hence

$$qvq^* = qvq^{-1} \in V.$$

Thus each $q \in \mathrm{Sp}(1)$ defines a linear map

$$\Theta(q) : v \mapsto qvq^* : V \to V,$$

giving a homomorphism

$$\Theta : \mathrm{Sp}(1) \to \mathrm{GL}(3, \mathbb{R}).$$

If $v \in V$ then

$$|v| = vv^* = x^2 + y^2 + z^2.$$

Now

$$\begin{aligned} |\Theta(q)v|^2 &= (qvq^*)(qva^*)^* \\ &= qvq^astqv^*q^ast \\ &= qvv^*q^* \\ &= vv^*qq^* \\ &= vv^* \\ &= |v|^2, \end{aligned}$$

since $vv^* \in \mathbb{R}$. Thus $\Theta(q)$ preserves the form $x^2 + y^2 + z^2$. Hence

$$\Theta(q) \in \mathrm{O}(3).$$

Since $\mathrm{Sp}(1) \cong S^3$ *is connected, so is* $\operatorname{im}\Theta(q)$. *Hence*

$$\Theta(q) \in \mathrm{SO}(3),$$

giving a homomorphism

$$\Theta : \mathrm{Sp}(1) \to \mathrm{SO}(3).$$

We have

$$\ker\Theta = \{q \in \mathrm{Sp}(1) : qv = vq \ \forall v \in V\}.$$

Since any quaternion is expressible as $Q = t1 + v$, *with* $t \in \mathbb{R}$, $v \in V$. *it follows that*

$$\ker\Theta = \{q \in \mathrm{Sp}(1) : qQ = Qq \ \forall Q \in \mathbb{H}\}.$$

It is readily verified that

$$Z\mathbb{H} = \mathbb{R} = \{t1 : t \in \mathbb{R}\}.$$

Hence

$$\ker\Theta = \{\pm 1\}.$$

To see that Θ *is surjective, ie* $\operatorname{im}\Theta = \mathrm{SO}(3)$, *we note that* $\mathrm{SO}(3)$ *is generated by half-turns* $\pi(l)$ *about an axis* l. *But it is readily verified that if* v *is a unit vector along* l *then*

$$\Theta(v) = \pi(v),$$

since $\Theta(v)$ *leaves* l *fixed, and*

$$v^2 = -vv^a st = -1,$$

and so

$$\Theta(v)^2 = I.$$

Hence Θ *defines a 2-fold covering of* $\mathrm{SO}(3)$.

(c) *Suppose*

$$\theta : G \to H$$

is a surjective homomorphism. Then a representation

$$\alpha : H \to \mathrm{GL}(V)$$

of H *in* V *defines a representation*

$$\alpha\theta : G \to \mathrm{GL}(V).$$

Furthermore, distinct representations of H give rise to distinct representations of G; and the representation $\alpha\theta$ is simple if and only if α is simple, since a subspace $U \subset V$ is stable under G if and only if it is stable under H.

Conversely, a representation

$$\beta : G \to \mathrm{GL}(V)$$

arises from a representation of H in this way if and only if

$$\ker\theta \subset \ker\alpha;$$

and if it does so arise, it is from a unique representation of H.

In the present case this shows that a representation of $\mathrm{SO}(3)$ arises from a representation α of $\mathrm{SU}(2)$ if and only if

$$\alpha(-I) = 1.$$

Looking at the definition of D_j by the action of of $\mathrm{SU}(2)$ on the space of homogeneous polynomials $f(z, w)$ of degree $2j$, we see that

$$f(-z, -w) = (-1)^{2j} f(z, w).$$

Thus

$$D_j(-I) = 1 \iff j \text{ is a half-integer.}$$

We conclude that the simple representations of $\mathrm{SO}(3)$ are the representations $D_0, D_1, D_2, \ldots$ of degrees $1, 3, 5, \ldots$.

(d)

6. Define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Determine the Lie algebras of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, and show that they are isomorphic.

   Are the groups isomorphic?

   Answer:

   (a)

7. Define the *exponential* e^X of a matrix $X \in \mathrm{Mat}(n, k)$, where $k = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Show that if X has eigenvalues λ, μ then e^X has eigenvalues e^λ, e^μ.

Which of the above 4 matrices X are themselves expressible in the form $X = e^Y$ for some real matrix Y? (Justify your answers in all cases.)

Answer:

(a) *The exponention of a square matrix is defined by*

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots.$$

(b) i. *If*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = I,$$

and so

$$\begin{aligned} e^X &= (1 + \frac{1}{2!} + \frac{1}{4!} + \cdots)I + (\frac{1}{1!} + \frac{1}{3!} + \cdots)X \\ &= \cosh(1)I + \sinh(1)X \\ &= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix} \end{aligned}$$

ii. *If*

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = -I,$$

and so

$$\begin{aligned} e^X &= (1 - \frac{1}{2!} + \frac{1}{4!} - \cdots)I + (\frac{1}{1!} - \frac{1}{3!} + \cdots)X \\ &= \cos(1)I + \sin(1)X \\ &= \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix} \end{aligned}$$

iii. *If*

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = I + Y,$$

where

$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$Y^2 = 0 \implies e^Y = I + Y = X,$$

and so

$$\begin{aligned} e^X &= e^I e^Y \\ &= \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} \end{aligned}$$

iv. *If*

$$X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = I - Y,$$

where

$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then

$$e^Y = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix}$$

from above, and so

$$e^{-Y} = (e^Y)^{-1} = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix}$$

and

$$\begin{aligned} e^X &= e^I e^{-Y} \\ &= \begin{pmatrix} e\cos 1 & e\sin 1 \\ -\sin 1 & e\cos 1 \end{pmatrix}. \end{aligned}$$

(c) *[Note that this part of the question only makes sense if* $k = \mathbb{R}$ *or* $\mathbb{C}$*. One does not in general speak of the eigenvalues or eigenvectors of a matrix* X *over* $\mathbb{H}$*, since the solutions of* $Xv = qv$ *will not in general form a subspace over* $\mathbb{H}$*.]*

Since e^X *is the same whether we consider* X *as a real or complex matrix, we may assume that* $X \in \mathrm{Mat}(n, \mathbb{C})$*.*

We know that in this case X can be triangulated, ie we can find T such that

$$TXT^{-1} = \begin{pmatrix} \lambda & c \\ 0 & \mu \end{pmatrix}$$

But then

$$TX^rT^{-1} = \begin{pmatrix} \lambda^r & c_r \\ 0 & \mu^r \end{pmatrix}$$

for each r, and so

$$Te^XT^{-1} = \begin{pmatrix} e^\lambda & c' \\ 0 & e^\mu \end{pmatrix}$$

Since Y and TYT^{-1} have the same eigenvalues, it follows that e^X has eigenvalues e^λ, x^μ.

(d) i. *From the last result,*

$$\begin{aligned} \det e^X &= e^\lambda\, e^\mu \\ &= e^{\lambda+\mu} \\ &= e^{\operatorname{tr} X}. \end{aligned}$$

In particular,

$$\det e^Y > 0$$

for all real Y.
Since

$$\det X = -1$$

in this case,

$$X \neq e^Y.$$

ii. *The map*

$$x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : \mathbb{C} \to \mathrm{Mat}(2, R)$$

is a homomorphism of $\mathbb{R}$-algebras under which

$$z \mapsto X \implies e^z \mapsto e^X.$$

The matrix

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

corresponds to the complex number i.
But $i = e^z$ where $z = \pi/2i$. Thus $X = e^Y$ where

$$Y = \begin{pmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{pmatrix}.$$

iii. We saw that

$$X = e^Y$$

in this case.

iv. As in the second case, the matrix X corresponds to the complex number

$$1 - i = \sqrt{2}e^{-i\pi/4}.$$

Thus $1 - i = e^z$ where

$$z = \log 2/2 - i\pi/4.$$

Hence $X = e^Y$, with

$$Y = \begin{pmatrix} \log 2/2 & \pi/4 \\ -\pi/4 & \log 2/2 \end{pmatrix}.$$

8. Show that the connected component G_0 of a linear group G is given by

$$G_0 = \{e^{X_1}e^{X_2}\cdots e^{X_r}\} \qquad (X_1, X_2, \ldots, X_r \in \mathscr{L}G),$$

where $r = 1, 2, \ldots$.

Explain how a representation α of a linear group G defines a representation $\mathscr{L}\alpha$ of $\mathscr{L}G$, and show that if G is connected then

$$\mathscr{L}\alpha = \mathscr{L}\beta \implies \alpha = \beta.$$

Sketch the proof that if G is simply connected then every representation of $\mathscr{L}G$ arises from a representation of G.

Answer:

(a) It is clear that G_0 is closed under multiplication; and it is closed under inversion, since

$$(e^{X_1}\cdots e^{X_r})^{-1} = e^{-X_r}\cdots e^{-X_1}.$$

Hence G_0 is a subgroup.

Also, G_0 is connected, since

$$T(t) = e^{tX_1}\cdots e^{tX_r} \quad (0 \le t \le 1)$$

is a path connecting I to $e^{X_1}\cdots e^{X_r}$.

Finally, G_0 is open. For there exists an open subset $U \ni 0$ in $\mathscr{L}G$ which is mapped homeomorphically onto an open subset $V = e^U \ni I$ in G; and

$$e^{X_1} \cdots e^{X_r} e^U$$

is an open neighbourhood of $e^{X_1} \cdots e^{X_r}$.

Since G_0 is an open subgroup, it is also closed; so G_0 and its complement are both open, and G_0 is the connected component of I in G.

(b) We assume the following result:

Lemma 1 *Suppose*

$$F : G \to H$$

is a continuous homomorphism of linear groups. Then there is a unique Lie algebra homomorphism

$$f : \mathscr{L}G \to \mathscr{L}H$$

such that

$$F(e^X) = e^{f(X)}$$

for all $X \in \mathscr{L}G$.

Now suppose

$$\alpha : G \to \mathrm{GL}(V)$$

is a representation of G.

By the Lemma, this gives rise to a Lie algebra homomorphism

$$\mathscr{L}\alpha : \mathscr{L}G \to \mathrm{gl}(V),$$

ie a representation of the Lie algebra $\mathscr{L}G$ in V, such that

$$\alpha(e^X) = e^{\mathscr{L}\alpha(X)}$$

for all $X \in \mathscr{L}G$.

(c) Suppose

$$\mathscr{L}\alpha = \mathscr{L}\beta = f,$$

say; and suppose $T \in G$. Then

$$T = e^{X_1} \cdots e^{X_r},$$

since $G_0 = G$. *Hence*

$$\begin{aligned}\alpha(T) &= \alpha(e^{X_1}) \cdots \alpha(e^{X_r}) \\ &= e^{fX_1} \cdots e^{fX_r} \\ &= \beta(e^{X_1}) \cdots \beta(e^{X_r}) \\ &= \beta(T).\end{aligned}$$

Thus

$$\alpha = \beta.$$

(d) *Suppose* G, H *are linear groups; and suppose the Lie algebra homomorphism*

$$f : \mathscr{L}G \to \mathscr{L}H$$

can be lifted to a homomorphism

$$F : G \to H,$$

satisfying

$$F(e^X) = e^{fX}$$

for all $X \in \mathscr{L}G$.

If

$$e^{X_1} \cdots e^{X_r} = 1$$

is an 'exponential relation' in G, *then*

$$\begin{aligned}e^{fX_1} \cdots e^{fX_r} &= F(e^{X_1}) \cdots F(e^{X_r}) \\ &= F(e^{X_1} \cdots e^{X_r}) \\ &= 1\end{aligned}$$

in H. *Thus*

$$e^{X_1} \cdots e^{X_r} = 1 \implies e^{fX_1} \cdots e^{fX_r} = 1.$$

Conversely if this is so, ie every exponential relation in G *maps to a corresponding relation in* H, *then the required homomorphism* $F : G \to H$ *can be defined as follows: given* $T \in G$, *suppose*

$$T = e^{X_1} \cdots e^{X_r}.$$

Then we set

$$F(T) = e^{fX_1} \cdots e^{fX_r}.$$

It follows at once from the hypothesis that $F(T)$ *is well-defined, ie independent of the 'exponential product' we choose for* T, *and that* F *is a homomorphism with* $\mathscr{L}F = f$.

i. *This property always holds locally: if all the partial products*

$$T = e^{X_1},\ e^{X_1}e^{X_2},\ e^{X_1}e^{X_2}e^{X_3},\ \ldots$$

lie in the logarithmic zone U then the corresponding relation in H holds.
It is sufficient to prove this for 'triangular relations'

$$e^X e^Y e^Z = 1.$$

This is established by showing that if X, Y, Z are small of size d then the 'discrepancy'

$$e^{fX} e^{fY} e^{fZ} - 1$$

is of order d^3.
Since a triangle of size d can be split into n^2 triangles of size d/n, it follows that the discrepancy of a triangle in U is in fact 0.

ii. *Now suppose G is simply connected, ie every loop is homotopically trivial.*

9. Define the Killing form $K(X, Y)$ of a Lie algebra $\mathscr{L}$.

   Determine the Killing form of $\mathrm{sl}(2, R)$, and show that it is non-singular.

   Show that if G is a compact linear group then the Killing form of G is negative definite or indefinite.

   Show conversely that if the Killing form a connected linear group G is negative definite then G is compact. Is the condition of connectedness necessary here?

   Answer:

   (a) *The Killing form is the symmetric bilinear form*

   $$K(X, Y) = \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y),$$

   where $\mathrm{ad}\, X$ *is the map*

   $$Z \mapsto [X, Z] : \mathscr{L} \to \mathscr{L}.$$

   (b) *We have*

   $$\begin{aligned} \mathrm{sl}(2, \mathbb{R}) &= \{X \in \mathrm{Mat}(2, \mathbb{R}) : \mathrm{tr}\, X = 0\} \\ &= \langle H, E, F \rangle, \end{aligned}$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} [H, E] &= HE - EH = 2E, \\ [H, F] &= HF - FH = -2F, \\ [E, F] &= EF - FE = H. \end{aligned}$$

Now

$$\begin{array}{lll} \operatorname{ad} H(H) = [H, H] = 0, & \operatorname{ad} H(E) = [H, E] = 2E, & \operatorname{ad} H(F) = [H, F] = -2F, \\ \operatorname{ad} E(H) = [E, H] = -2E, & \operatorname{ad} E(E) = [E, E] = 0, & \operatorname{ad} E(F) = [E, F] = H, \\ \operatorname{ad} F(H) = [F, H] = 2F, & \operatorname{ad} F(E) = [F, E] = -H, & \operatorname{ad} F(F) = [F, F] = 0. \end{array}$$

Thus $\operatorname{ad} H, \operatorname{ad} E, \operatorname{ad} F$ *take matrix forms*

$$\operatorname{ad} H \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \; \operatorname{ad} E \mapsto \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \operatorname{ad} F \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

with respect to the basis H, E, F *of* $\mathrm{sl}(2, \mathbb{R})$.
Hence

$$\begin{aligned} K(H, H) &= \operatorname{tr}((\operatorname{ad} H)^2) = 8, \\ K(E, E) &= \operatorname{tr}((\operatorname{ad} E)^2) = 0, \\ K(F, F) &= \operatorname{tr}((\operatorname{ad} F)^2) = 0, \\ K(H, E) &= \operatorname{tr}(\operatorname{ad} H \operatorname{ad} E) = 0, \\ K(H, F) &= \operatorname{tr}(\operatorname{ad} H \operatorname{ad} F) = 0, \\ K(E, F) &= \operatorname{tr}(\operatorname{ad} E \operatorname{ad} F) = 4. \end{aligned}$$

Thus the Killing form (with respect to the basis H, E, F*) is given by the quadratic form*

$$K(x, y, z) = K(xH + yE + zF, xH + yE + zF) = 8x^2 + 8yz,$$

which is non-singular (but neither positive nor negative).

(c) *Suppose* G *is compact.*
The adjoint representatiion of G *in* $\mathscr{L}G$ *is given by*

$$\operatorname{ad} g(X) = gXg^{-1}.$$

Since G is compact, there is an positive-definite quadratic form left invariant under this action. Choose coordinates so that this form is

$$x_1^2 + \cdots + x_n^2.$$

Then

$$\operatorname{ad} g \in \mathrm{O}(n).$$

This representation of G corresponds to the adjoint representation of $\mathscr{L}G$:

$$\operatorname{ad} X(Z) = [X, Z].$$

It follows that

$$\operatorname{ad} X \in \mathrm{o}(n) = \{S \in \mathrm{Mat}(n, \mathbb{R}) : S + S' = 0\}.$$

But then if $\operatorname{ad} X = S$

$$(\operatorname{ad} X)^2 = -S'S,$$

and so

$$\operatorname{tr}((\operatorname{ad} X)^2) = -\operatorname{tr}(S'S) \leq 0,$$

with equality only if $S = 0$.

We conclude that $K(X, Y)$ *is negative definite or indefinite.*

(d) *If the Killing form* $K(X, Y)$ *of a linear group G is negative-definite then it* does *follow that G is compact. The proof is long, and the following is more of an overview. [A complete proof can be found in Chapter 12 of Part IV of my notes.]*

We assume the following result:

Lemma 2 *The Killing form of a compact group G is left invariant by G:*

$$K(\operatorname{ad} g(X), \operatorname{ad} g(Y)) = K(X, Y).$$

If the Killing form is negative-definite, we can choose coordinates so that it takes the form

$$-(x_1^2 + \cdots + x_n^2).$$

Thus if we set

$$G_1 = \operatorname{im} \mathrm{Ad},$$

then

$$G_1 \subset \mathrm{O}(n).$$

[In fact, since G is connected,

$$G_1 \subset \mathrm{SO}(n).]$$

Now

$$\ker \mathrm{Ad} = ZG,$$

the centre of G*. This is discrete; for*

$$\mathscr{L}(ZG) = Z(\mathscr{L}G) = 0,$$

since

$$\begin{aligned} X \in ZG &\implies \mathrm{ad}\, X = 0 \\ &\implies K(X, Y) = 0 \end{aligned}$$

for all $Y \in \mathscr{L}G$*, in which case the Killing form will be singular. Thus*

$$Ad : G \to G_1$$

is a covering.
A compactness argument shows that ZG *is finitely-generated. If it is finite we are done; it is easy to see that G is compact. If not then* ZG *is an infinite discrete abelian subgroup; Thus*

$$ZG = T \oplus \mathbb{Z}^n$$

where T *is finite and* $n > 0$*.*
In particular, there is a non-trivial homomorphism

$$\chi : ZG \to \mathbb{R}.$$

We are going to show that this can be extended to a homomorphism

$$X : G \to \mathbb{R}.$$

This will lead to a contradiction. For the kernel of the corresponding Lie algebra homomorphism will be an $(n-1)$*-dimensional subspace of* $\mathscr{L}G$*; and it is easy to see that this subspace would in fact be an ideal, whose complement with respect to the non-singular Killing form would be a 1-dimensional ideal* $I \subset \mathscr{L}G$*. This would necessarily be trivial, so that* $K(X, Y) = 0$ *for all* $X \in I$*,* $Y in \mathscr{L}G$*, contradicting the assertion that* $K(X, Y)$ *is negative-definite.*

To construct the extension X, we note first that a standard compactness argument shows we can find a compact subset $C \subset G$ such that

$$\mathrm{Ad}(C) = G_1.$$

Now let $u(x)$ be a function on G with compact support such that

$$x \in C \implies u(x) > 0.$$

Set

$$w(x) = \frac{u(x)}{\sum_{z \in ZG} u(zx)}$$

Then

$$w(x) > 0 \text{ and } \sum_{z \in ZG} w(zx) = 1$$

for all $x \in G$. (We can think of $w(x)$ as a kind of weight on G, allowing us to smooth out the given homomorphism χ.)
Set

$$f(x) = \sum_{z \in ZG} w(zx)\chi(z).$$

Suppose $z' \in ZG$. As z runs over ZG, so does zz'. Hence

$$\begin{aligned} f(x) &= \sum_{z \in ZG} w(zz'x)\chi(zz') \\ &= \sum_{z \in ZG} w(zz'x)(\chi(z) + \chi(z')) \\ &= f(z'x) + \chi(z'). \end{aligned}$$

In other words,

$$f(zx) = f(z) - \chi(z).$$

Thus if we define the function $F : G \times G \to \mathbb{R}$ by

$$F(x, y) = f(xy) - f(x),$$

then

$$F(zx, y) = F(x, y).$$

But that means we can regard F as a function on $G_1 \times G$, where $G_1 = G/ZG$ is compact. This allows us to integrate over G_1, and set

$$X(g) = \int_{G_1} F(g_1, g).$$

It is a straightforward matter to verify that this function $X : G \to \mathbb{R}$ is in fact a homomorphism extending χ.

[This is a very complicated argument. Hopefully nothing as horrid as this will be asked in the exam!]