A DISSERTATION SHEWING, THAT THE ACCOUNT OF THE DOCTRINES OF FLUXIONS AND OF PRIME AND ULTIMATE RATIOS DELIVERED IN A TREATISE, ENTITLED, A DISCOURSE CONCERNING THE NATURE AND CERTAINTY OF SIR ISAAC NEWTON'S METHOD OF FLUXIONS, AND OF PRIME AND ULTIMATE RATIOS, IS AGREEABLE TO THE REAL SENSE AND MEANING OF THEIR GREAT INVENTOR

By

Benjamin Robins

(The Present State of the Republick of Letters, April 1736, pp. 290–335)

Edited by David R. Wilkins

2002

NOTE ON THE TEXT

This dissertation appeared in *The Present State of the Republick of Letters* for April 1736. It was reprinted (with some alterations) in *Mathematical Tracts of the late Benjamin Robins Esq.*, edited by James Wilson, London, 1761.

Two corrections were given in the Errata printed at the end of the *May* 1736 issue of the *Republick of Letters*. These are as follows:

the sentence commencing with the words 'Moreover in some observations he made on a Letter of *Leibnitz* ...' was added, following the words '*nominare licet*, &c.';

the first occurrence of '*Bernoulli*' was changed to '*John Bernoulli*'.

The following spellings, differing from modern British English, are employed in the original 1736 text: streight, sollicitous, falsly, fixt, rendred, center, varyed, weakning, expresly, compleat.

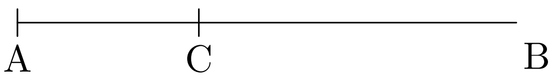
David R. Wilkins
Dublin, June 2002

A Dissertation *shewing, that the account of the doctrines of Fluxions, and of prime and ultimate ratios, delivered in a treatise, entitled,* A discourse concerning the nature and certainty of Sir *Isaac Newton*'s methods of fluxions, and of prime and ultimate ratios*, is agreeable to the real sense and meaning of their great inventor. By* B. Robins, F.R.S.

[*The Present State of the Republick of Letters*, April 1736, pp. 290–335.]

The principal pursuit of the geometers of the last century was in search after general methods for discovering the nature and properties of curved figures. Sir *Isaac Newton* succeeded so happily in this attempt, as to establish a very extensive method of computation for these purposes. And being dissatisfied with the doctrine of indivisibles, or of infinitely small quantities, which had been hitherto employed in these enquiries, he introduced his doctrines of fluxions, and of prime and ultimate ratios, as juster principles, whereon to found this method of computation. The book, we are now going to consider, is confined to the explanation of these principles.

The substance of what we have there said with relation to the conception and nature of fluxions, is as follows.



Let the line A B be supposed to be tracing out by the motion of a point setting forward from A; then the velocity of that point in any part of its motion, as at C, will be the Fluxion of the line A C at that time described.

And as the velocity of this point in different parts of its motion may be supposed to be any how increased or diminished, the degree of this increase or diminution at C is the second fluxion of the line A C.

Again, since this increase or diminution may be of different degrees at different places, it may itself also have a change, which at different places will be greater or less. And the degree of this change at the place C is the third fluxion of the line A C.

The fluxion of other quantities is not the velocity of the motion, whereby they are increased, but the rate of that increase. And here the terms velocity, celerity, and degree of swiftness, which originally belong to actual motion, being applied to this rate of increase in a sense somewhat figurative, Sir *Isaac Newton* in lines chose to call the actual motion, wherewith they are described, the fluxion of those lines, as being an idea more obvious than the rate of their increase, which otherwise might have been assigned for the fluxion of lines, as well as of all other quantities.

The case of lines being thus the simplest in the doctrine of fluxions, we have shewn in our discourse, how that of all other flowing quantities may be reduced to this by causing a point so to pass over any streight line, that its length measured out, while the other flowing quantity is describing, shall augment in the same proportion with such flowing quantity. So that the fluxion, or velocity of increase of this fluent, will be ever proportional to the actual velocity of the point describing the line. And from this consideration we demonstrated in particular, that when a curvilinear space is supposed to be described by the uniform and parallel motion of an ordinate, the fluxion of that space will be every where as the length of the ordinate.

That our account of fluxions is the same with that given by Sir *Isaac Newton* will appear from his own words. In the introduction to his treatise of Quadratures he says, *Quantitates mathematicas, non ut ex partibus quam minimis constantes, sed ut motu continuo descriptas hic considero. Lineæ describuntur, ac describendo generantur, non per appositionem partium, sed per motum continuum punctorum, superficies per motum continuum linearum, &c. Considerando igitur, quod quantitates æqualibus temporibus crescentes, & crescendo genitæ, pro velocitate majori vel minori qua crescunt ac generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum, quibus generantur; & has motuum vel incrementorum velocitates nominando fluxiones, & quantitates genitas nominando fluentes, &c.* Again, in the book itself he says, *Quantitates indeterminatas, ut motu perpetuo crescentes vel decrescentes, id est, ut fluentes vel defluentes, in sequentibus considero, designoque literis* z, y, x, v, *& earum fluxiones, seu celeritates crescendi noto iisdem literis punctatis. Sunt & harum fluxionum fluxiones, sive mutationes magis aut minus celeres, quas ipsarum* z, y, x, v *fluxiones secundas nominare licet, &c.* Moreover in some observations he made on a Letter of *Leibnitz*, which were printed at the end of *Raphson*'s *Historia Fluxionum*, he declares, *That in a paper written by him so long ago as* 1665, *the direct method of Fluxions was set down in these words: An equation being given expressing the relation of two or more lines* x, y, z, &c. *described in the same time by two or more moving bodies* A, B, C, &c. *to find the relations of their velocities* p, q, r, &c. And in the Philosophical Transactions, N° 342. he says, *When he considers lines as fluents described by points, whose velocities increase or decrease, the velocities are the first fluxions, and their increase the second.* And this has particular regard to the first part of his description of fluxions, where he calls them *velocitates motuum.*

Here it most manifestly appears, that our description of fluxions is the very same, Sir *Isaac Newton* has himself delivered; and as he has never attempted to represent them in any other light, we cannot sufficiently admire, how he came to be charged by a late writer with having given so various and inconsistent accounts of them.* And this notion of fluxions and their different Orders is evidently free not only from any impossible, but even obscure suppositions. Insomuch as this writer, for the support of his objections against this doctrine, found it necessary to represent the idea of fluxions as inseparably connected with the doctrine of prime and ultimate ratios, intermixing this plain and simple description of fluxions with the terms used in that other doctrine, to which the idea of fluxions has no relation: and at the same time by confounding this latter doctrine with the method of *Leibnitz* and the foreigners, has proved himself totally unskill'd in both.

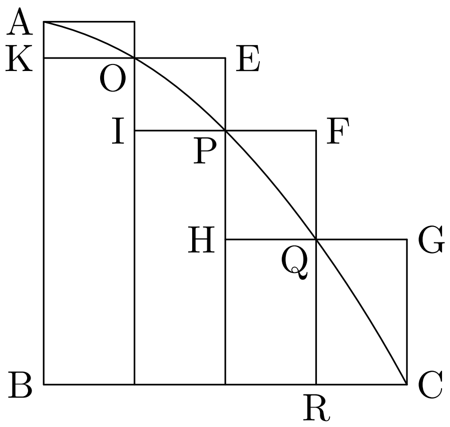
These two methods of Sir *Isaac Newton* are so absolutely distinct, that their author had formed his idea of fluxions before his other method was invented, and that method is no otherwise made use of in the doctrine of fluxions, than for demonstrating the proportion

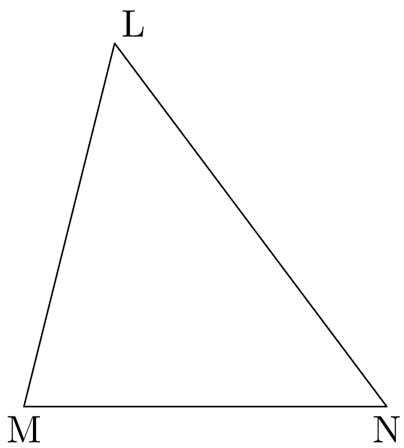
* Defense of Free-Thinking in Mathematicks, *p.* 41.

between different fluxions. For, in Sir *Isaac Newton*'s words*, as the fluxions of quantities are nearly proportional to the contemporaneous increments generated in very small portions of time, so they are exactly in the first ratio of the *augmenta nascentia* of their fluents. With regard to this passage the writer of the Analyst has made a twofold mistake. First, he charges Sir *Isaac Newton*, as saying these fluxions are very nearly as the increments of the flowing quantity generated in the least equal particles of time. Again, he always represents these *augmenta nascentia*, not as finite indeterminate quantities, the nearest limit of whose continually varying proportions are here called their first ratio, but as quantities just starting out from non-existence, and yet not arrived at any magnitude, like the infinitesimals of the differential calculus. But this is contrary to the express words of Sir *Isaac Newton*, who after he had shewn how to assign by his method of prime and ultimate ratios the proportion, that different fluxions have to one another, he thus concludes. *In finitis autem quantitatibus Analysin sic instituere, & finitarum nascentium vel evanescentium rationes primas vel ultimas investigare consonum est geometriæ veterum: & volui ostendere, quod in methodo fluxionum non opus sit figuras infinite parvas in geometriam introducere.*

And this leads us to consider Sir *Isaac Newton*'s doctrine of prime and ultimate ratios, which is of the greatest use in demonstrations relating to curves. It is no other than an abbreviation and improvement of the form of demonstrating used by the ancients on the like occasions. For this reason we premised to our explanation of prime and ultimate ratios a short description of that method, which we shall now consider more at large.

As no two different curves can be so laid on each other as to coincide either in whole or in part, it is evident, that the spaces bounded by such curve lines cannot be immediately compared either with each other, or with right-lined figures; nor for the same reason can such Spaces be the Sums or differences or others, that are capable of being thus compared. In examining then the dimensions and proportions of these curvilinear spaces, some other method must be made use of, than those that are required in the comparison of right-lined figures.





Suppose a space bounded by the curve A C, and the right lines A B, B C. Let the base

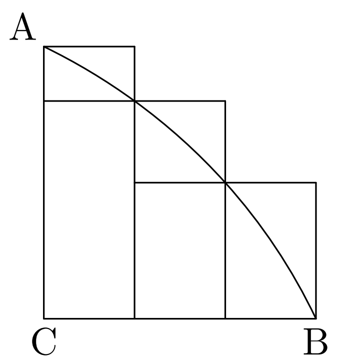
* Fluxiones sunt quam proxime ut fluentium augmenta æqualibus temporis particulis quam minimis genita, &, ut accurate loquar, sunt in prima ratione augmentorum nascentium. *Newton. Introd. ad Quad.*

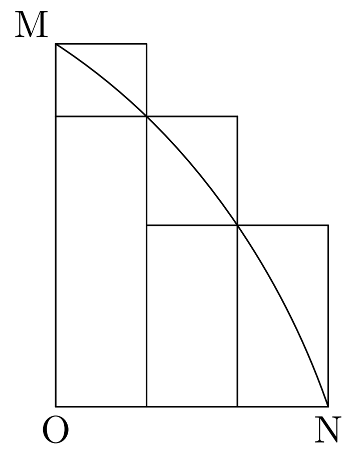
B C be divided into any number of parts, and on those divisions let parallelograms be drawn forming the figures A D O E P F Q G C B and K O I P H Q R B, the first circumscribing, and the last inscribing the given space A C B. It is now obvious, that by diminishing the breadth of these parallelograms, these figures may be made to differ from that space A C B, and from each other by less than any space, how minute soever, that shall be named; that is, the circumscribed figure can be made less than any space greater than the curve, and the inscribed figure greater than any space less than the curve.

If by considering the properties of these inscribed and circumscribed figures, which arise from the nature of the curve, they are adapted to, a right lined space L M N can be assigned, that shall be greater than every inscribed figure, and less than every circumscribed figure, this right-lined space L M N may be proved to be equal to the curvilinear space A C B.

For were it greater, a circumscribed figure could be made less; and if it were less, an inscribed figure could be made greater.

Instead of both inscribed and circumscribed figures, we might have made use of one of them only, suppose of the inscribed, by proving the space M L N to be greater than every inscribed figure, and also capable of being approached by such a figure within less than any given difference. For thus the space M L N can neither be less nor greater than the curve. If it were less, an inscribed figure would exceed it; and if it were greater, no inscribed figure could approach it so near as its excess above the curve.





Again, suppose there are two curvilinear spaces, A C B, and M O N. If parallelograms, whose breadth may be any how diminished, are drawn inscribing and circumscribing these curves; and if they are described in such a manner, that the circumscribed figure of one curve to the circumscribed figure of the other, and the inscribed to the inscribed, has one and the same constant proportion in every description: I say, that the curve A C B is to the curve M O N in that proportion, which the inscribed and circumscribed figures constantly bear to each other.

For no space greater than A C B can have to M O N this proportion; since if it could, a figure might be circumscribed about A C B less than this supposed greater space, and this circumscribed figure to its correspondent figure circumcribing M O N would be in the same proportion, as the supposed greater space to the curve M O N; that is, four quantities being in the same proportion, the first would be less than the third, and the second greater than the fourth. Nor can any space less than A C B have to M O N the constant proportion of the figures in one curve to the figures in the other. For if it could, a figure might be inscribed within A C B, which would be greater than this supposed lesser space; and this inscribed figure to its correspondent figure inscribing M O N would be in the same proportion, as this imagined lesser space to the curve M O N; that is, four quantities being in the same proportion, the first would be greater than the third, and the second less than the fourth. Thus no space but A C B can be to M O N in the constant proportion of the circumscribed and inscribed figures.

If the proportion of the circumscribed figures, or of the inscribed is not the same in every description, but constantly changing (suppose one of the ratios perpetually increasing, and the other diminishing) as the breadths of the parallelograms are contracted; then it will be shewn by a similar process, that the ratio, which is greater than every increasing proportion, and less than every diminishing proportion, will be the true ratio of the curvilinear spaces.

The demonstration may here too proceed by the inscribed or circumscribed figures only, as in the first example, if they bear a constant unchangeable ratio in every description, or if the ratio can be found, which the ratio of those figures does perpetually approach, and to which by diminishing the parallelograms it will at last come nearer than to any other ratio that shall be given.

Though we have here made use of parallelograms, yet any other method of describing the circumscribed and inscribed figures, may equally take place, provided the figures arising from such description can be made to differ from the curve by less than any difference whatever, that shall be proposed: And in the description of these figures lies the great artifice of these demonstrations; for they ought to be so drawn, that the right lined space in our first instance, greater than the inscribed, and less than the circumscribed figure, may from the consideration of these figures be most easily determined; or that the proportions of these figures in the following instances, may in every description be easily assignable.

In the manner here described did the ancient geometers demonstrate whatever they discovered relating to the dimensions or proportions of curve lines, curvilinear spaces, and solids bounded by curved surfaces. And this is the form of demonstration, which is now called the method of exhaustions. But as these demonstrations, by determining distinctly all the several magnitudes and proportions of these inscribed and circumscribed figures, did frequently extend to very great lengths, other methods of demonstrating had been contrived, whereby to avoid these circumstantial deductions. The first attempt of this kind, known to us, is that we mentioned to have been made by *Lucas Valerius*. But afterwards *Cavalerius*, an *Italian*, about the year 1635, advanced his method of indivisibles, in which he proposes not only to abbreviate the ancient demonstrations, but to remove the indirect form of reasoning used by them of proving the equality or proportion between lines and spaces from the impossibility of their having any different relation; and to apply to these curved magnitudes the same direct kind of proof, that was before applied to right lined quantities.

This method of comparing magnitudes invented by *Cavalerius*, supposes lines to be compounded of points, surfaces of lines, and solids of planes; or, to make use of his own description, surfaces are considered as cloth consisting of parallel threads, and solids are considered as formed of parallel planes, as a book is composed of its leaves, with this restriction, that the threads or lines, of which surfaces are compounded, are not to be of any conceivable breadth, nor the leaves or planes of solids of any thickness. He then forms these propositions, that surfaces are to each other as all the lines in one to all the lines in the other, and solids in like manner in the proportion of all their planes.

But this method of indivisibles, as here explained, is manifestly founded on inconsistent and impossible suppositions. For while the lines, of which surfaces are supposed to be made up, are real lines of no breadth, it is obvious, that no number of them can form the least imaginable surface: if they are supposed to be of some sensible breadth, in order to be capable of filling up spaces, that is, in reality to be parallelograms, how minute soever be their altitude, the surfaces may not be to each other in the proportion of all such lines in one to all the like lines in the other; for surfaces are not always in the same proportion to each other with the parallelograms inscribing them.

The same contradictory suppositions do obviously attend the composition of solids by parallel planes, or of lines by such imaginary points.

This heterogeneous composition of quantity, and confusion of its species, so different from that distinctness, for which the mathematicks were ever famous, was oppposed at its first appearance by several eminent geometers, particularly by *Guldinus* and *Taquet*, who not only excepted to the first principles of this method, but tax the conclusions formed upon it, as erroneous. But as *Cavalerius* took care that the threads or lines, of which the surfaces to be compared together were formed, should have the same breadth in each (as he himself expresses it) the conclusions deduced by his method might generally be verified by sounder geometry; since the comparison of these lines was in effect the comparing together the inscribed figures.

As in the application of this method, error by proper caution might be avoided, the assistance it seemed to promise in the analytical part of Geometry, made it eagerly followed by those, who were more desirous to discover new propositions, than sollicitous about the elegance or propriety of their demonstrations. Yet so strange did the contradictory conception appear of composing surfaces out of lines, and solids out of planes, that in a short time it was new modelled into that form, which it still retains, and which now universally prevails amongst the foreign mathematicians.

In this reformed notion of indivisibles, surfaces are now supposed as composed not of lines, but of parallelograms, having infinitely little breadths and solids in like manner as formed of prisms having infinitely little altitudes. By this alteration it was imagined, that the heterogenous composition of *Cavalerius* was sufficiently evaded, and all the advantages of his method retained.

But here again the same absurdity occurs, as before. For if by the infinitely little breadth of these parallelograms we are to understand, what these words literally import, that is, no breadth at all; then they cannot, any more than the lines of *Cavalerius*, compound a surface; and if they have any breadth, the right lines bounding them cannot coincide with a surface bounded by a curve line.

The followers of this new method grew bolder than the followers of *Cavalerius*; and having transformed his points, lines, and planes, into infinitely little lines, surfaces, and solids, they pretended, they no longer compared together heterogeneous quantities, and insisted on their principles being now become genuine: but the mistakes, they frequently fell into, were a sufficient confutation of their boasts. For not withstanding this new model, the same limitations and cautions were still necessary. For instance, this agreement between the inscribing figures and the curved spaces, to which they are adapted, is only partial, and in applying their principles to propositions already determined by a juster method of reasoning, they easily perceived this defect: both in surfaces and solids it was evident at the first view, that the perimeters disagreed. But as no one instance can be given, where these indivisible or infinitely little parts to so compleatly coincide with the quantities they are suppos'd to compound, as in every circumstance to be taken for them without producing erroneous conclusions; we find where a surer guide to correct their reasoning was wanting or disregarded, these figures were often imagin'd to agree, where they ought to have been suppos'd to differ.

We might produce numerous instances of such errors from the writings of modern computists. The most celebrated of these, *Leibnitz* and *John Bernoulli*, will furnish us with sufficient examples, from the single error of supposing infinitely small arches absolutely to coincide with their chords.

The first, in two dissertations, one on the resistance of fluids, and another on the motion of the heavenly bodies, has on this principles reasoned falsly concerning the lines intercepted between curves and their tangents, as has been observed by Sir *Isaac Newton* himself. And *Bernoulli*, in a dissertation likewise on the resistance of fluids, has made the same mistake, and even insisted over Sir *Isaac Newton*, where he had by means of his juster method of reasoning avoided that error; and *Bernoulli* upon the same principles has erred a second time in a pretended solution of the problem concerning isoperimetrical curves.

But the proceeding of Mons. *Parent* is so very extraordinary, that it deserves also to be mentioned. He has had the rashness to oppose erroneous deductions from this absurd principle to the most indubitable demonstrations of the great *Huygens*. It having been shewn by *Galileo*, that any heavy body will descend through the chord of a circle terminating at its lowest point in the same time, as it will fall through its perpendicular diameter; from this principle of indivisibles, that the arch and the chord do at last coincide, it was falsly concluded, that the time of the fall through the smallest arches must be equal to the time of the fall through the diameter. This contest between demonstration and error has been thought of such importance, that several elaborate dissertations have been published to shew by what means the idolized doctrine of infinitely small quantities could produce fallacious conclusions*.

Thus it appears, that the doctrine of indivisibles contains an erroneous method of reasoning, and in consequence thereof, in every new subject, to which it shall be applied, is liable to fresh errors.

It is also manifest, that the great brevity it gave to demonstrations, arose entirely from the absurd attempt of comparing curvilinear spaces in the same direct manner as right lined figures can be compared; for in order to conclude directly the equality or proportion of such spaces no scruple was made of supposing, contrary to truth, that rectilinear figures capable of such direct comparison could adequately fill up the spaces in question; whereas the doctrine of exhaustions does not attempt from the equality or proportion of the inscribing or circumscribing figures to conclude directly the like proportions of these spaces, because those figures can never in reality be made equal to the spaces they are adapted to. But as these figures may be made to differ from the spaces, to which they are adapted, by less than any space proposed, how minute soever, it shews, by a just though indirect deduction, from these inscribing and circumscribing figures, that the spaces, whose equality is to be proved,

* See Memoires de l'Acad. des Sciences. 1722.

can have no difference, and that the spaces, whose proportion is to be shewn, cannot have a different proportion from that assigned them.

But Sir *Isaac Newton*, by his doctrine of prime and ultimate ratios, has found out the proper medium, whereby to avoid the impossible notion of indivisibles on the one hand, and the length of exhaustions on the other. As it was Sir *Isaac Newton*'s express design absolutely to free his method of reasoning from every part of the obscurity and inconsistent notions of indivisibles, and as all the objections raised against him have entirely been grounded on a suspicion, that he has not fully succeeded in that design; we thought the readiest method of vindicating this doctrine was to cast it into a form as remote as possible from any appearance of agreement with that absurd system.

For this purpose we defined the ultimate magnitude of any varying magnitude to be a fixt quantity, which the varying one can approach within any degree of nearness, and yet can never pass: And the ultimate ratio of any varying ratio to be a fixt ratio, which the varying one can approach with any degree of nearness, but yet can never pass.

These definitions thus premised, we demonstrated the two following propositions.

1. That when the varying magnitudes continue always in the same ratio, their ultimate magnitudes will be also in the same ratio.

2. That all the ultimate ratios of the same varying ratio are the same to each other.

In this form it was easy to explain the doctrine of prime and ultimate ratios without the use of any of those expressions, which had been misunderstood to have reference to the method of indivisibles. But to make appear the identity of this doctrine, as by us expounded, with what Sir *Isaac Newton* has himself deliver'd, we premised a short representation of the true sense, in which we apprehended his phraseology ought to be understood. As he founded his doctrine on the first lemma in the first section of his *Principia Philosophiæ*, in my discourse I have thus represented his meaning in that lemma, That in this method any fixt quantity or ratio, which some varying quantity or ratio by a continual augmentation or diminution shall perpetually approach, but never pass, is considered as the quantity or ratio, to which the varying one will at last or ultimately become equal; provided the varying quantity or ratio can be made in its approach to the other to differ from it less than by any difference how minute soever, that can be assigned. Consonant to this representation of Sir *Isaac Newton*'s meaning, in the account given of my book, this lemma was thus interpreted; those quantities are to be esteemed ultimately equal, and those ratios ultimately the same, which are perpetually approaching each other in such a manner, that any difference how minute soever being given, a finite time may be assigned, before the end of which the difference of those quantities or ratios shall become less than that given difference.

And to this interpretation the following remark was subjoin'd, That this lemma did not mean, or necessarily imply, that any point of time was assignable, wherein these varying magnitudes would become actually equal, or the ratios really the same; but only that no difference whatever could be named, which they should not pass.

This interpretation of Sir *Isaac Newton* is evidently conformable to our definitions.

A learned gentleman, who, concealed under the name of *Philalethes Cantabrigiensis*, had expressly entered into controversy with the Author of the *Analyst*, though he allows the truth of our method of reasoning to be unquestionable; yet he thinks we have in some measure deviated from the exact intention of Sir *Isaac Newton*. He has interpreted the forementioned lemma after a manner something different. But his interpretation does not ascribe to the word *given*, used by Sir *Isaac Newton* in this lemma, the true sense of that word in Geometry, but supposes it to stand for *assignable*; whereas it properly signifies only what is actually assigned. *Philalethes* insinuates that by our interpretation, and the forementioned remark upon it, Sir *Isaac Newton* is rendred obnoxious to the charge of first supposing what he would prove, and with proving only what he has before supposed. But our interpretation cannot possibly mean less than this, that those quantities and ratios will have no last difference, which are perpetually approaching each other in such a manner, that any difference how minute soever being given, a finite time may be assigned, before the end of which the difference of those quantities or ratios shall become less than that given difference. This is certainly no identical proposition, tho' its truth be very obvious; and Sir *Isaac Newton*'s demonstration of it is accordingly very short.

However I acknowledge, had Sir *Isaac Newton* here no other intention, but simply to prove so obvious a truth, a distinct proposition for this purpose only might perhaps have well been spared. But we must consider this proposition in another light. There are two ways whereby good writers explain the use of terms they introduce: one is by expressly defining them; another, when, to avoid that formality, they convey the sense of such terms by their manner of using them. And to make appear, that Sir *Isaac Newton*, by the demonstration annexed to this lemma, has sufficiently evinced, in what sense the lemma itself must be understood, and at the same time to prove what that sense is, it was shewn, that this demonstration is no less applicable to quantities, which only approach without limit to the ratio of equality, than it will agree to such quantities, as at last become actually equal. For this purpose this demonstration was applied to the ordinate of an hyperbola, compared with the same continued to the asymptote, which do approach without limit to the ratio of equality, though they never become actually equal. But as *Philalethes* has taken exception to this instance, not conceiving how to regulate this approach, so as to bound it within a finite time; without enquiring how far that limitation was necessary to our purpose, we shewed a method of adding this circumstance by causing a line to turn upon the center of the hyperbola, and pass with an equable motion from the diameter to the asymptote: for by supposing the forementioned ordinate continually to accompany the intersection of this line with the hyperbola, the whole motion here required will be brought within such a finite space of time, as he imagin'd necessary. And in this view it is equally manifest, that the ordinate, and its continuation, can never become equal, till they are both extended to infinity, and all idea of them is lost.

By this we think it very evident, that Sir *Isaac Newton* has neither demonstrated the actual equality of all quantities capable of being brought under this lemma, nor that he intended so to do.

Whenever the quantities or ratios compared in this lemma are capable of an actual equality, they must really become so. But when they are incapable of such equality, the phrase of ultimately equal must of necessity be interpreted in a somewhat laxer sense; that is, as Sir *Isaac Newton* in the 71 proposition of the first book of his *Principia* expresses it, *pro æqualis habeantur, are to be esteemed equal*, and means only that such quantities or ratios approach without limit. Accordingly we find, that immediately after this lemma he uses the expressions *ultimo in ratione æqualitatis*, and *ultimo æquales*, as synonymous terms. However, as in every subject of this lemma all ultimate difference is excluded, the consequences drawn from it are equally just and perspicuous, whether the quantities do or do not become actually equal, and the ratios actually coincident. And this restriction of the sense of this lemma is absolutely necessary to be attended to in this doctrine, because Sir *Isaac Newton* himself has applied it to quantities and ratios incapable of an actual equality or agreement.

In the account of our Discourse, the lemma immediately following, where parallelograms are inscribed, and others circumscribed to a curvilinear space, was produced as an example of this. It was there also observed, that vanishing quantities may never actually have that proportion, which, according to this lemma, is said ultimately to belong to them.

In this second lemma Sir *Isaac Newton* directs, that the number of these parallelograms should be augmented *in infinitum*. This must not be interpreted, till the number become infinitely great: for this is the express language of indivisibles. We render the words *in infinitum, endlessly*, and perform, what is here directed, by that simple and obvious method practised by the ancient geometers, of continually subdividing the base of the curve. And it is manifest, that such subdivision can never be actually finished. *Philalethes* on the other hand endeavours to describe a complex kind of motion, whereby he apprehends this multiplication of the parallelograms can be brought to a period. But as the inscription and circumscription of these figures require, that the bases of those parallelograms be constantly equal, and each some aliquot part of the whole base, any such description by continued motion is necessarily excluded, as has been already observed*. *Philalethes* charges this with being too hasty an assertion; because in another lemma, namely the third, parallelograms are supposed to be described to a curve, whose bases should not be equal. But it was not asserted, that Sir *Isaac Newton* had supposed this equality in all the propositions, wherein he may have had occasion to consider parallelograms described to curves; but that it was constantly and always to take place in this second lemma; which being a distinct and separate proposition must have a demonstration compleat within itself.

However, should we so far depart from Sir *Isaac Newton*, as to admit of this complex kind of motion, what idea can we form of the inscribed or circumscribed figure, to which we are at last actually to arrive, which with any propriety of speech is to be styled equal to the curve?

Sir *Isaac Newton* in the corollaries annexed to the third lemma expressly declares, there is no ultimate sum of these parallelograms, nor no ultimate figure compounded of them, distinct from the very curve itself. *Philalethes* himself acknowledges this. Though how far this concession agrees with the rest of his opinion will be best understood hereafter. But at present we shall examine more particularly Sir *Isaac Newton*'s meaning in the corollaries now mentioned.

The treatise of Sir *Isaac Newton*, in which this doctrine is delivered, is written throughout with that degree of brevity, as occasions a general complaint of the difficulties attending the study of it. And this conciseness is no where perhaps more remarkable than in the section now under consideration; insomuch, that it often requires careful attention to discover the exact meaning, and full force of the expressions. More than once Sir *Isaac Newton*, to convey his intention the easier to those, who had been accustomed to the method of indivisibles, has introduced some expressions analogous to the phraseology of that doctrine, when the brevity, he had prescribed to himself, occasioned his not giving express notice of it. Of this kind

* Republick of Letters for last December, pag. 444.

we must reckon the conclusion of the following passage. *Ultimæ rationes illæ, quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant; et quas propius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, neque prius attingere quam quantitates diminuuntur in infinitum**. The last words of this passage mean in reality no more, than that the quantities will never have the ratio mentioned; and the only reason to be assigned, why Sir *Isaac Newton* expressed this in the manner he has done, is, that he addresses himself to those, who had been accustomed to indivisibles, and accommodates himself to their language. The expression in the first of the corollaries before mentioned is evidently adapted to the same design. The *ultima summa* there mentioned, in strict propriety of speech, has no kind of meaning, for it is really infinite. His intention could only be here to signify, that what in the language of indivisibles might be called the last sum of these parallelograms, or the figures supposed in that method to be composed of such infinite number of parallelograms, or other right-lined figures, is really nothing distinct from the curve itself.

To assert that any collection of these inscribed or circumscribed parallelograms can ever become actually equal to the curve, is certainly an impropriety of speech; for equality can properly subsist only between figures distinct from each other. Such expressions therefore are so far from giving any additional advantage to this method of reasoning, that they can only tend to confound this method with indivisibles. For we have already shewn, that the essence of indivisibles consists in endeavouring to represent to the mind such inscribed or circumscribed figure, as actually subsisting, equal to the curve. That method does not merely depend on a particular set of exceptionable expressions; but however the phraseology be varyed, yet while the same mode of thinking is attempted, we are still involved in that erroneous doctrine. Whatever state of the inscribed or circumscribed figure is supposed previous to this equality, or by whatever changes it is conceived to degenerate into its imagined last form: yet as long as we fancy ourselves capable of seeing directly in this last form those properties which these figures had before, we are still immerged in the doctrine of indivisibles. It is certainly therefore an advantage to this method, to seclude from it any expressions leading towards such faulty conceptions.

But *Philalethes* seems apprehensive of weakning the force of Sir *Isaac Newton*'s demonstrations by this means, when he tells us, that *Sir Isaac Newton contents not himself with any the most near approximations, but carries his demonstrations to the utmost accuracy and geometric rigor; accordingly every one of the examples he has given in the lemmata of the first section, are of such quantities and ratios, as do actually arrive at their respective limits.* Does *Philalethes* here suppose the truth of Sir *Isaac Newton*'s demonstrations to depend on this actual equality of the variable quantity and its limit? He confesses our demonstrations to be just, which do not suppose this actual equality.

He also says, that *the supposing this actual equality seems greatly to exceed the method of the ancients in perspicuity as well as in the conciseness of its demonstrations.*

That this method should be more perspicuous is impossible, the method of the ancients being perfect in that respect. Certainly there are not in their method, what *Philalethes* (though I think without reason) insinuates of this, any demonstrated truths, that must be

* Newtoni Princip. pag. 38.

owned, though we do not perfectly see every step, by which the thing is brought about. That it exceeds the method of the ancients in conciseness is true; but that is not occasioned by this supposed actual equality of the variable quantity and its ultimate; since, as we have shewn this to be at the best but a superfluous circumstance, the supposing it necessary is far from promoting conciseness, that it adds to the length of this doctrine, by obliging us to labour in every case after some idea of motion, however intricate, whereby to represent to our minds this actual equality. In the case we have been considering, the motion proposed for forming these parallelograms is sufficiently intricate. If it were convenient to make use of a polygon described within or without a curve, as in the first proposition of the second section of the first book of the *Principia*, another kind of motion must be accommodated to that case. And so for every variation of these inscriptions and cirumscriptions we must strain our imagination for some involved and perplexed kind of motion applicable to each.

The sum of what we have been saying amounts to this, that in our interpretation of Sir *Isaac Newton*, no subtle inquiry after means to bring about an actual equality between the curve and the inscribed and circumscribed figures is at all necessary; which is affixing intricate circumstances to this method, no way necessary to the truth or clearness of the demonstrations. For this actual equality can scarce be represented to the mind, but in such complex, not to say confused ideas, that will bring us upon the very borders of indivisibles, and render us perpetually obnoxious to the absurdities of that doctrine. It is therefore without question an advantageous representation of this method, to free it from every perplexity of this kind. And we have shewn our interpretation, which thus removes this doctrine quite beyond the reach of every objection, that has hitherto been levelled against it, to be conformable to the most proper signification of Sir *Isaac Newton*'s words, both in his second lemma, where those figures are considered, and also in the first, where the principles of this method are established.

And our interpretation of this first lemma is still more abundantly necessary in applying this lemma to what Sir *Isaac Newton* calls vanishing quantities. For as these quantities are supposed continually to diminish, and by that means to have their proportions varied; nothing is more evident, than that their diminution will never bring them actually to bear that ratio to one another, with which in this lemma the ratio of these quantities is said to become ultimately the same.

This will most evidently appear by Sir *Isaac Newton*'s words. *Ultimæ rationes illæ, quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites, ad quos quantitatum sine limite decrescentium rationes semper appropinquant; & quas proprius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, neque prius attingere, quam quantitates diminuuntur in infinitum. Res clarius intelligetur in infinite magnis. Si quantitates duæ, quarum data est differentia, augeantur in infinitum, dabitur harum ultima ratio, nimirum ratio æqualitatis; nec tamen ideo dabuntur quantitates ultimæ, seu maximæ, quarum ista est ratio. In sequentibus igitur, si quando facili rerum conceptui consulens, dixero quantitates quam minimas, vel evanescentes, vel ultimas; cave intelligas, quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite.*

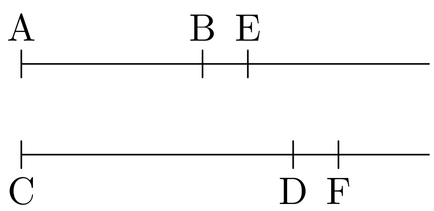
Here it is expresly declared, that these *ultimæ rationes quibuscum quantitates evanescunt*, are not *rationes quantitatum ultimarum*; but only limits, to which the ratios of these quantities, which themselves decrease without any limit, continually approach; and to which these ratios can come within any difference, that may be given, but never pass, nor even reach those limits, before the quantities are diminished to nothing. To explain this more distinctly, and to prevent, as much as possible, his reader from seeking after any state or condition, at which these quantities can actually arrive, wherein to be the subjects of this proportion; he draws a parallel between this case, and the case of quantities supposed to augment without end. For says he, such quantities may have an ultimate ratio, though here will not be any last quantities as the subjects of that ratio.

Philalethes is very unwilling to allow this intended for an exact parallel. But Sir *Isaac Newton* expresly affirms, his intention of introducing it was to render more clear the thing, he had been immediately speaking of, that is, the nature of the ultimate ratios of quantities decreasing without limit.

Philalethes indeed asserts a difference between the two cases. In vanishing quantities he asserts their ratio actually to come up to (*attingere*) their limit; but in the other not. It is true the quantities in one case may be reduced absolutely to nothing, but in the other can never be extended to an infinite magnitude. Yet in both cases the quantities are equally incapable of being converted by the variation ascribed to them into any condition, wherein they will be the subjects of that ratio, which is called their ultimate. Nay, Sir *Isaac Newton* has been so particularly careful, lest any of his readers, from brief expressions, which they would afterwards find in this book, should imagine, he in the least favoured the attempt of indivisibles, to pursue such quantities to a diminution actually infinite, here further adds this express caution, that by whatever name he might hereafter denote these vanishing quantities, they were never to be consider'd as determinate, but as variable ones, diminishing without limit; consonant to what he said before, *Nolim indivisibilia, sed evanescentia divisibilia intelligi.**

This interpretation of Sir *Isaac Newton*, so expresly conformable to his own words, at once dissipates all the objections the author of the *Analyst* has raised against the demonstrations, whereby Sir *Isaac Newton* proves the operations in his method of fluxions, though condemned with so much freedom for fallacious and inconclusive.

The form of those demonstrations may be represented thus.



Let the two points B and D, one tracing out the line A B, and the other the line C D, be supposed to set forward together from A and C, and to arrive in the same time at B and D. Now it is required, from the given relation between these described lines A B and C D, in all the correspondent magnitudes of those lines, to determine the proportion of the velocities of the points at B and D; that is, to determine the proportion of the fluxions of those lines A B and C D from the known relation of the flowing lines.

Suppose these points advance to E and F in the same time. Then if their velocities are always in the same proportion, the augments B E and D F will be in the proportion of those velocities. But if one of the points, suppose D, is more accelerated than the other B; then the

* Newton. Princip. p. 37.

ratio of D F to B E will be a proportion greater than that of the velocities: but the smaller B E and D F are taken, the nearer will the proportion of these spaces D F and B E approach to the proportion of the velocities at D and B; and the difference between these two proportions may be diminished in any degree whatever by a sufficient diminution of the augments D F and B E. Then the ultimate proportion of the decreasing quantities D F and B E will be the true proportion of the velocities, or of the fluxions at D and B.

This is expresly the method, by which Sir *Isaac Newton* determines the proportion of the fluxions of different magnitudes in all cases. Having first compared such contemporaneous augments as have a finite, that is, a real magnitude; then he supposes these augments continually to diminish, and having determined the nearest proportion, to which they constantly tend during their diminution, he assigns this as the true proportion of the velocities or fluxions.

Sir *Isaac Newton* describes the process, we have here explained, by these words:* *Evanescant jam augmenta illa & eorum ultima ratio erit, &c.*

Now this passage is thus translated and commented on by the author of the *Analyst*, *Let these increments vanish*, i. e. *let the increments be nothing*; and from thence he takes great pains to shew the absurdity of comparing together, and assigning the proportion of quantities, after they are supposed to be destroyed.

But here he commits a twofold error: first, in imagining, that the operation, which the increments are by Sir *Isaac Newton* supposed to undergo in order to have their ultimate proportion assigned, and which he describes by the verb *evanescant*, is confined to that point of time only, at which the increments are actually gone and abolished; and secondly, in imagining, that by the ultimate ratio of varying quantities is meant a ratio, that those quantities do at some time or other exist under.

As to the first supposition, when the points E and F are conceived to move backwards, till they arrive at B and D, the diminishing of the augments B E and D F, as well as their abolition at last in the points B and D effected by this means, is by Sir *Isaac Newton* comprehended under the general description of *evanescant*, let them vanish, as is most evident, not only by the passage above quoted from Sir *Isaac Newton*'s introduction to his treatise of Quadratures, where he says his analysis investigated *finitarum nascentium vel evanescentium rationes primas vel ultimas*, but also by the words since produced, that *the vanishing quantities by him considered are divisible, and not determinate, but continually diminishing.*

Since therefore these vanishing quantities are expresly declared by Sir *Isaac Newton* to be finite and variable, his expression in this place must be understood to relate to the whole time they are vanishing. And his words are free from any impropriety; for the term vanishing is daily applied to objects during the time of their disappearing, before they are actually out of sight, absolutely signifying no more than going to vanish. Just as we say the sun is setting in the most limited signification of that word, so soon as its under limb touches the horizon; and as soon as ever the sun is quite out of sight, it is no longer setting, but actually set: so these quantities, being of a finite, that is, of a real magnitude, do not vanish instantaneously, but with the utmost propriety may be said to be vanishing all the time they are undergoing the diminution ascribed to them.

The second error of the author of the *Analyst*, that of supposing the ultimate ratio of

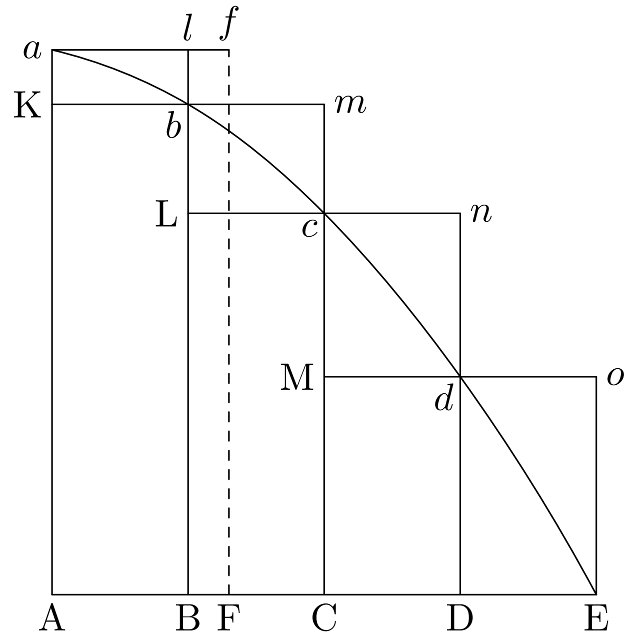
* Introd. ad Quad. p. 44.

varying quantities to be a ratio, which these quantities must some time or other exist under, we have fully shewn to be contrary to Sir *Isaac Newton*'s express declaration.

Upon this mistake is grounded the charge of indivisibles being unavoidably supposed in this doctrine, when he says, no quantity can be admitted as a medium between a finite quantity and nothing, without admitting infinitesimals. By what has been above said it appears, that *Philalethes* had no necessity, for avoiding the consequence here charged upon the doctrine, to have recourse to that definition of a *nascent increment*, which follows. *A nascent increment is an increment just beginning to exist from nothing, or just beginning to be generated, but not yet arrived at any assignable magnitude, how small soever.*

Here a nascent augment seems to be represented, as a quantity neither of any finite magnitude, nor yet as absolutely nothing, and can scarce be conceived of otherwise, than as in an intermediate *state* between both. I apprehend *Philalethes* was induced to frame this definition from the terms *nascent* and *evanescent*, by which those, who had composed demonstrations, or writ upon this subject in our language, had rendred Sir *Isaac Newton*'s words *nascentes* and *evanescentes*. And these *English* words bearing the form of nouns adjective, they too frequently neglected the addition of prime and ultimate, necessary to render the sense compleat in expressing the ratio, which is the limit to the ratios of these quantities. We find Sir *Isaac Newton* so cautious in this particular, that in the account of the *Commercium Epistolicum* published in the *Philosophical Transactions*, though written by him in *English*, he retains the *Latin* expressions. Since Sir *Isaac Newton* intended by *quantitates nascentes* and *evanescentes* finite and indeterminate quantities capable of bearing different proportions, the term prime or ultimate is absolutely necessary to express that ratio, which is the limit of those different ones.

From this inadvertent use of the words *nascent* and *evanescent* we expressed a dislike to them in our Discourse.



But now, to sum up the whole of what we have said upon this head, since Sir *Isaac Newton* has expresly told us, that the quantities, he calls *nascentes* and *evanescentes*, are by him always considered as finite quantities; that ratio called their prime or ultimate cannot be the ratio, which those quantities themselves at any time must actually have. Our interpretation therefore of the lemma so often mentioned is absolutely necessary in applying it to these quantities. And as we have shewn no different sense to be required in any other subject of this doctrine; so our representation of Sir *Isaac Newton*'s mind has not only been proved to be conformable to the genuine meaning of his words, but is also perfectly consistent with itself: whereas we must confess ourselves at a loss to reconcile *Philalethes* with himself in the acknowledgment he makes, that the ultimate form even of the perimeter of the inscribed figure in Sir *Isaac Newton*'s second and third lemmas is no other than the curve itself; that is, in each triangle $a\mathrm{K}b$, $b\mathrm{L}c$, $c\mathrm{M}d$, $d\mathrm{D}E$, the rectilinear sides $a\mathrm{K}$, $\mathrm{K}b$, $b\mathrm{L}$, $\mathrm{L}c$, $c\mathrm{M}$, $\mathrm{M}d$, $d\mathrm{D}$, DE, vanish into the curve itself: if at the same time he supposes evanescent quantities subsisting at each point of the curve, which can be the subjects of the proportions between the ordinate, tangent, and subtangent.

We shall now proceed to consider, what Sir *Isaac Newton* has called the *momenta* of quantities. This term was used very early by him. About the year 1666 he drew up a short Discourse *de Analysi per æquationes numero terminorum infinitas.* Here the word moment frequently occurs.* He has told us *this tract teaches how to resolve finite equations into infinite ones, and how by the method of moments to apply equations both finite and infinite to the solution of problems.* He says, that *he there called the moment of a line a point in the sense of* Cavalerius*, and the moment of an area a line in the same sense.* The passage in the book, to which this relates, is as follows. *Nec vereor loqui de unitate in punctis, sive lineis infinite parvis, siquidem proportiones ibi jam contemplantur geometræ, dum utuntur methodis indivisibilium.* And he has told us, *from the moments of time he gave the name of moments to the momentaneous increases, or infinitely small parts of the abscissa and area generated in moments of time.* He says, *Leibnitz hath no symbols of fluxions in his method, but used the symbols of moments or differences* dx, dy, dz. All this is suitable to the doctrine of indivisibles. He likewise tells us, *because we have no ideas of infinitely little quantities, he introduced fluxions into his method, that it might proceed by finite quantities as much as possible.* Hence it appears, he had not at the first discovered his doctrine of prime and ultimate ratios, which entirely rejects indivisibles, or infinitely little quantities; but at length falling upon it, *he founded his method* [of fluxions] *on the* primæ quantitatum nascentium rationes, *which have a being in geometry, whilst indivisibles, upon which the differential method is founded, have no being either in geometry, or in nature.* Accordingly he tells us, *When he is demonstrating any proposition, he used the letter* o *for a finite moment of time, or of its exponent, or of any quantity flowing uniformly, and performs the whole calculation by the geometry of the ancients in finite figures or schemes without any approximation: and so soon as the calculation is at an end, and the equation is reduced, he supposes, that the moment* o *decreases* in infinitum*, and vanishes. But when he is not demonstrating, but only investigating a proposition, for making dispatch he supposes the moment* o *to be infinitely little, and forbears to write it down, and uses all manner of approximations, which he conceives will produce no error in the conclusion.* Here Sir *Isaac Newton* declares, he was wont to use the word moment in two senses; examples of both which he then mentions. And it is observable in his rule for finding the relation of fluxions, as published out of his old papers by Dr. *Wallis* is 1693, the

* Philos. Trans. No. 342.

word moment is used in the sense of indivisibles, but when he came to give that rule himself in his book of *Quadratures* first printed in 1704, he used that word in the other sense.

Before he had published any thing on these subjects, he thought fit for the sake of brevity to introduce this term *moment* in the second book of his *Principia Philosophiæ*. As the geometers of his time had been much accustomed to indivisibles, he did not scruple there to describe moments according to the sense of that doctrine, as he had done formerly, to be *incrementa vel decrementa momentanea*. As in another place of that treatise he acknowledges his using several expressions favouring indivisibles, but at the same time shews how that idea may be corrected, when such expressions occur; so likewise here he does the like: he shews how to correct the idea arising from this description of moments. He says, *you must never consider their magnitudes, but their prime ratio.* He adds, *it would come to the same thing, if instead of these moments you used the velocities of increase or decrease of quantities, which he is wont to call fluxions, or if you used any finite quantities proportional to these fluxions.*

Notwithstanding all this caution of Sir *Isaac Newton*, he has not escaped being censured. I therefore endeavoured in my discourse to clear up this affair of the moment, not thereby to vindicate the genuineness of Sir *Isaac Newton*'s methods of fluxions, and of prime and ultimate ratios, which I had before sufficiently shewn to be accurate, and did not in the least depend on the interpretation of the word moment, but to make appear, as on this head there had been raised great clamor and boasting, that it was without any manner of foundation. To this end in my Discourse I gave a description of moments suitable to the doctrine of prime and ultimate ratios; and since Sir *Isaac Newton*'s demonstration of the moment of a rectangle had been attempted to be exploded, though it is most accurate as well as brief, this is likewise explained; and is also more fully enlarged upon in the account of my book.

The mistakes, that have here arisen, were occasioned by not sufficiently attending to Sir *Isaac Newton*'s last mentioned caution. From thence it will appear, in calling $\frac{1}{2}a$ and $\frac{1}{2}b$ the halves of the moments of A and B, by a and b he meant finite quantities in the prime or ultimate ratio of the correspondent increments or decrements of A and B.

Upon this principle, if the sides of a rectangle, which are denoted by A and B, be augmented and diminished by half such lines expressed by a and b, as shall be in the ultimate ratio of the increments or decrements of the sides A and B, generated in equal portions of time; the difference $(a\mathrm{B} + b\mathrm{A})$ of such rectangles, as are contained by the sides A and B thus augmented and diminished, will express the momentum of the original rectangle.

The exception to the demonstration, *Philalethes* has given, of the method for finding the *momentum* of a rectangle is, that as it demonstrates too much, it must of necessity be inconclusive.

He has endeavoured to prove, that the moment of the rectangle is an arithmetick mean proportional between the contemporaneous increment and decrement of the same rectangle.

But it has been shewn, that this is only true, when the sides augment in the same constant proportion.

Consequently the supposed demonstration of *Philalethes* must be defective; for there is no part of it, but the conclusion, that contains any restriction to this particular case.

Philalethes says, that *Sir* Isaac Newton *never admitted of indivisibles, nor of quantities infinitely small, conceived as actually existing in a fixed determinate and invariable state.*

That Sir *Isaac Newton* has made use of indivisibles in the very sense of *Cavalerius*, and that the doctrine of moments was originally founded on them, we have already proved from his own words.

We are also told, *that Sir* Isaac Newton *in the lemma, where he determines the moments, amongst other methods of conceiving them considers these moments as the differences of* Leibnitz, or as *Philalethes* afterwards explains it, *as finite quantities exceedingly small.* But this is directly contrary to Sir *Isaac Newton*'s description, who, speaking of these *momenta*, as *incrementa momentanea*, in the sense of indivisibles, says *particulæ finitæ non sunt momenta.* Afterwards indeed he adds a caution, whereby we may understand the signification of the word moment in the true sense of his method of prime and ultimate ratios; that is, that these *momenta* may be expressed by finite quantities, not confined to be exceedingly small, but of any magnitude, provided they were in the prime or ultimate ratios of their corresponding increments or decrements.

And it is afterwards said, that *the course taken by Sir Isaac Newton to find the finite difference of variable quantities, though not rigorously geometrical in the higher cases, yet approaches nearer to geometric rigour than the method of* Leibnitz.

Now I say, that were these differences in this sense considered as finite small quantities; however Sir *Isaac Newton*'s computation might come nearer the value of any quantity sought after, than that of *Leibnitz*; yet considered as a medium of demonstration to determine the absolute value of such quantity, both will be totally, and therefore equally, void of geometric rigour.

It is said, that *in the first case of this lemma Sir* Isaac Newton *is naturally to be understood as considering the sides of the rectangle to flow either uniformly or proportionally.*

I say this cannot be the natural interpretation of that case, because it is immediately quoted to prove the second case, where the augmentation is confessedly different. Nor can there be any reason assigned to shew, why it should be thus understood: for Sir *Isaac Newton* has computed the moment of the rectangle, not by supposing the sides increased and diminished by their respective increments and decrements generated in equal portions of time, but by finite lines expressing half their correspondent moments, as we explained it above; so that his determination is by this means general, and according to the utmost geometric rigour.

I hope, I have here not only shewn, that the account, I lately published, of Sir *Isaac Newton*'s doctrines of fluxions, and of prime and ultimate ratios is entirely conformable to the sense of that great man; but have also placed them in such light, as the objections, that have been raised against them, will at once appear to proceed from misconceptions and misrepresentions of these subjects. Sir *Isaac Newton* has been charged with having given various and inconsistent accounts of his methods, and been represented as struggling with insuperable difficulties, and imposing on his followers. How little reason there is for all these imputations will be manifest from the following considerations.

Sir *Isaac Newton* being very young at the beginning of his mathematical Studies, discovers a very extensive and compendious method of calculation, which he readily applied to the finding the *maxima* and *minima*, drawing tangents, determining the curvature of curves, squaring curvilinear surfaces, and to other problems of the like sort. About the year 1665, because, as he says, we have no ideas of indivisibles, or infinitely little quantities, he introduced fluxions into his calculations, that he might proceed without indivisibles, as much as possible. But in determining the proportions of these fluxions he still allowed himself some use of infinitely little quantities. No doubt, but upon reading the ancients he from thence would have been enabled to have demonstrated the proportions of fluxions according to their accurate methods; for he did much more, by finding out one of his own, more compendious than theirs, and equally geometrical. This served not only to demonstrate the proportions of fluxions, but was applicable to the synthetic demonstration of all propositions relating to curves. When he discovered this method of prime and ultimate ratios, we cannot certainly know. We are sure he had part of it in 1669, on account of a demonstration added at the end of his *Analysis per æquationes, &c.* which was sent at that time by Dr. *Barrow* to Mr. *Collins.* But most probably he had not then compleated this method, since in the *Lectures* he read the same year at *Cambridge* on his admirable discoveries in opticks, he used indivisibles in his demonstrations.

It was in 1686 he first disclosed his doctrine of prime and ultimate ratios in his immortal work the *Mathematical Principles of Natural Philosophy.* It is surprising with what modesty, and as it were fearfulness to offend such, as had been admirers of indivisibles, he introduced so excellent and truly geometrical method, by censuring the other in the softest manner. Tho' in answering the objections, that might be started against his own method, he evidently proves, he was fully apprised of the real imperfections of indivisibles, at the same time shewing a way to avoid them; yet he scarce condemns them himself, and frequently makes use of expressions peculiar to them, thinking it sufficient once for all to inform those, who did not approve of indivisibles, how to correct such expressions, and render them conformable to his method of prime and ultimate ratios.

In this treatise he but once mentions his doctrine of fluxions, and tho' what he says of them is short, yet it is very just. It seems as if he took notice of them chiefly, that a cypher, he thought fit to explain relating to them, might be understood. He here indeed demonstrates synthetically, and very accurately the foundation of his method of calculation, which is common to the methods of fluxions, of prime and ultimate ratios, of moments, and of differentials.

In the year 1704. he published his book of Quadratures, a work worthy his profound genius. He had now sufficiently seen the abuses, that had been made of the doctrine of infinitely small quantities, in what was called the differential calculus. In the introduction to this book he delivers a very distinct account of his method of fluxions, and teaches how to find out their proportions by his method of prime and ultimate ratios, in order, as he says, to shew there was no occasion in the use of fluxions to introduce infinitely little quantities into geometry: but still with his usual modesty saying errors might be avoided in the other method, if we proceeded cautiously. In all this plain narrative of matter of fact, there appears no inconsistency in Sir *Isaac Newton*'s accounts of his methods, or the least shadow of having been ever puzzled or confounded in his ideas about them.

Hence it is very manifest, that as Sir *Isaac Newton* used at first indivisibles, so he soon corrected those faulty notions by his doctrine of fluxions, and afterwards by that of prime and ultimate ratios. And all the absurdity of expression, and all the inconsistency with himself charged on him by the author of the *Analyst*, arises wholly from misrepresentation. For example, it has been asserted, that there is as little sense in the phrase, a fluxion of a fluxion, as to speak of the velocity of a velocity*. This objection supposes, that the simple word velocity can always be substituted in the room of the word fluxion. But by Sir *Isaac*

* Defense of Free-thinking in mathematicks, *p.* 24.

Newton's description of the fluxions of magnitudes, it is evident, that the single words can never be used promiscuously: for the fluxion of any quantity is not the velocity of that quantity, but the velocity, wherewith it at all times augments or diminishes; for instance, the fluxion of a line is not the velocity, wherewith that line moves, but the velocity of the point, by whose motion the line is described. Therefore, first fluxions not being the velocities of their fluents, but the velocities, with which the fluents increase or diminish, the fluxion of a fluxion is not the velocity of the first fluxion, but the velocity or degree of swiftness, with which the first fluxion increases or diminishes. Again, because Sir *Isaac Newton* has said fluxions are in the first ratio of the *augmenta nascentia*; therefore fluxions, and what some are pleased to call nascent augments, are so absurdly confounded together, that the expressions of a fluxion of a fluxion and the nascent augment of a nascent augment are represented as synonymous. Lastly, the description, Sir *Isaac Newton* has given, of moments for their use, who had been accustomed to the method of indivisibles, is set up as a standard to interpret his doctrines of fluxions and of prime and ultimate ratios; and the cautions he gave, in order that the term moment might be understood suitable to those doctrines, have been either neglected or misunderstood, and himself represented as imposing on his followers. And as Sir *Isaac Newton* from the demonstration, he has given, of the *momentum* of a rectangle, and of other more compound quantities deduces the *momentum* of a power, which *momentum* may be applied to determine the fluxion of such powers: he is thence charges with being unsatisfied with the truth of his demonstration, nay of exerting the utmost subtilty and skill, the greatness of his genius was capable of, in struggling with a difficulty imagined insuperable; for no other reason, than because he has given another more direct demonstration of the fluxion of powers in the introduction to his treatise of Quadratures, which differing a little in form from the other, is represented as grounded on different principles; whereas in the demonstration of the *momentum* of a rectangle the moments of the sides are taken instead of their compleat increments upon the very same foundation, as the superfluous terms are rejected out of the increment of the power in the other demonstration. And this author thinks it very reasonable to suspect, that Sir *Isaac Newton* might not be fully persuaded of the truth of what he has undertaken expresly to demonstrate, because he has happened to declare himself so cautious upon a certain point, as to decline the attempt of demonstrating it, though he was strongly persuaded of its truth.*

In the account of his book of Quadratures given in the *Acta Eruditorum* in 1708, there is an insinuation reflecting on the candour and honour of this most excellent person.

This occasioned old papers to be examined in order to vindicate his reputation; the result of which was published in a treatise entituled *Commercium Epostolicum D. J. Collins, &c.* This made it plainly appear, that he was the real inventor of these methods, we have been describing, and proved his rival *Leibnitz* in several particulars a most egregious plagiary.

In an account of this book written by Sir *Isaac Newton* himself, and often referred to in this dissertation, he without reserve discovered his dislike to indivisibles or infinitely little quantities, and being here obliged to compare his methods with that called *Leibnitz*'s differential calculus, he absolutely denied this latter capable of demonstrating a proposition, because it was founded on indivisibles; which clearly proves, he lookt on no demonstration as valid, that was built on those absurd principles.

* Analyst, *p.* 27.

To conclude, I am sorry, that in any particulars relating to Sir *Isaac Newton*'s doctrine of prime and ultimate ratios I should differ in sentiment from the learned *Philalethes*, and perhaps from several other excellent persons. As I learnt this doctrine solely from Sir *Isaac Newton*'s Works; so the account, I have published of it, is agreeable to the opinion, I had ever entertained concerning it. Notwithstanding the terms of indivisibles, and other figurative expressions, that frequently occur in his writings, I always thought his true design very manifest. About many of these forms of speech he has given us a caution in order to prevent mistakes, declaring he used them to assist the imagination. But there are others of the like sort not expresly taken notice of, he perhaps believing no body liable to be deceived by their use. The ultimate ratio of vanishing quantities, the ratio with which quantities vanish, are in strict propriety of speech figurative expressions: nay, the last form of a figure, and the form, wherewith a figure vanishes, might be interpreted upon the foot of indivisibles. But here these phrases only signify the limits, to which the ratios of the vanishing quantities, and the forms of the changing figures approach within any degree of nearness without being ever able to arrive at them. As in Opticks, though the focus of rays after refraction is understood to denote the place, where they meet; yet the point on the axis of any spherical glass, assigned by optical writers for such focus, is the point, which only limits all the intersections of a pencil of rays with the axis after their refraction, and is so far from being a point, where any number of rays actually unite, that no individual ray, excepting that which passes in the axis itself, is supposed to pass through it. Thus the last from of a vanishing figure is not the form, which that changing figure will ever arrive at, or subsist under; but a form, to which it will approach within any degree of nearness possible to be assigned. To assert that a vanishing triangle, for instance, would ever exist under that form, which is called its last, would be, as the author of the *Analyst* expresses it, to imagine a triangle in a point. But this is not Sir *Isaac Newton*'s meaning. These ultimate ratios, and last forms are the limits only of the continually varying ratios, and changing forms of the vanishing quantities and figures. And it is the whole purpose of this method, from these varying ratios and changing forms to find the fixt ratio and stable form, which is said to be the last of these varying ones.

This at present is my real opinion of this doctrine. I do not deny, but some expressions may be found in Sir *Isaac Newton*'s works consistent with another sense; and I am also very sensible, how frequently he accommodated himself to the way of writing of the mathematicians of his time, partly for assisting their imagination, and partly, perhaps, for fear he might seem to affect innovation. But I make no scruple of interpreting these expressions suitably to my representation of this doctrine; for otherwise I acknowledge myself totally incapable of reconciling this method of prime and ultimate ratios with the character, the author himself has given of it.

He prefers it to indivisibles, because, he says, they have no place in geometry, or in nature; and he often insists, that his method is conformable to the geometry of the ancients. But how can we pursue the variable and fleeting forms of the inscribing and circumscribing figures *in infinitum*, so that, when they should become equal to the curve, they may not totally withdraw themselves from the imagination, and all idea about them be lost? It is certain such refined metaphysical notions are not in the least analogous to that simplicity and clearness in the ideas, to which the ancient geometers ever confined themselves. However, the conception of vanishing quantities ever actually arriving at their ultimate proportion, has above been shewn to be unquestionably inconsistent with nature. Nay farther, I will not pretend on any other principles, than those I have set forth in my discourse, to defend Sir *Isaac Newton* from the consequences of the mistakes, he has been charged with: for the asserting, that the varying ratios of vanishing quantities, and the changing forms of vanishing figures would ever attain those ultimate ratios, and last forms, seems to me directly to impinge upon those principles, he has expressly declared to be absurd.