A REVIEW OF SOME OF THE PRINCIPAL OBJECTIONS THAT HAVE BEEN MADE TO THE DOCTRINES OF FLUXIONS AND ULTIMATE PROPORTIONS

By

Benjamin Robins

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NOTE ON THE TEXT

This article appeared in *The Present State of the Republick of Letters* for December 1735. It was reprinted (with some alterations) in *Mathematical Tracts of the late Benjamin Robins Esq.*, edited by James Wilson, London, 1761.

In the first figure, the vertex labelled 'f' is labelled 'c' in the original article. This would seem to be a typographical error: another of the vertices of the diagram is also labelled 'c', and moreover this figure is intended to reproduce that accompanying *Lemma II*, *Lib I* of Newton's *Principia*, where this vertex is labelled 'f'. The figure here is copied from the article by Jurin published the previous month.

The following spellings, differing from modern British English, are employed in the original 1735 text: synonimous, center.

David R. Wilkins
Dublin, June 2002

A Review *of some of the principal Objections that have been made to the Doctrines of Fluxions and Ultimate Proportions; with some Remarks on the different Methods that have been taken to obviate them.*

[*The Present State of the Republick of Letters*, December 1735, pp. 436–447.]

The objections that have been made to the conception and nature of Fluxions, have principally arisen, either from confounding this doctrine with the method of indivisibles, and the differential calculus of foreigners, or from supposing (as fluxions are said to be velocities) that the fluxion of a quantity, and the velocity of a quantity, were synonimous terms; forgetting that it is not to the quantities themselves, but to their degree of increase or decrease that this velocity intended by the Fluxion is ascrib'd. But as these mistakes can be no longer made without the greatest negligence or disingenuity, it may be reasonably supposed, that no exception of this kind will for the future be insisted on. We shall therefore at this time confine ourselves to the objections of another kind, such as have been urged against those operations, by which the proportion of the fluxions of different flowing quantities are determined.

These objections have been particularly levell'd at that expression of Sir *Isaac Newton, Fluxiones sunt in ultima ratione decrementorum evanescentium vel prima nascentium.* Which being usually thus translated, that fluxions are in the ultimate ratio of the evanescent decrements, or in the first ratio of the nascent augments, it has from hence been ask'd, What these nascent or evanescent augments are? If of any magnitude, then it will be confess'd by the espousers of this doctrine, their ratio is not the same with the ratio of the fluxions. If it is answered, that they are of no magnitude; it is then said, that to talk of a ratio of nothings, is such a strain of language, as it is supposed the warmest followers of the inventor will scarce undertake to defend.

To obviate this objection, two explanations have been given of this quotation.

The first endeavours to shew how this imagin'd difficulty may be avoided, not by considering these evanescent decrements, and nascent augments as being actually vanished, in which case they can have no proportion, nor yet as being of any real magnitude, when their proportion cannot be the same with the proportion of the fluxions, but by supposing that there can be represented to the mind some intermediate state of these augments or decrements at the very instant in which they vanish.

Another writer has endeavoured to shew, that this objection is founded on an erroneous hypothesis: for that by the ultimate proportion of varying quantities was only meant the limit of their varying proportion, and not a proportion that these varying quantities could ever exist under during their variation; and consequently that the true explication of this passage should be, Fluxions are in that proportion, which is the ultimate to all those varying proportions that the decrements bear to each other, whilst they are vanishing or diminishing; that is, the limit of the proportion that the decrements bear to each other as they diminish, is the true proportion of the fluxions. By this interpretation, which is supported by Sir *Isaac*

Newton's own Words, the above-mentioned objection immediately falls to the ground; since it is altogether founded on the supposition, that the decrements in their imagin'd evanescent state did really bear to each other the proportion of the fluxions; whereas this passage, when truly understood, does not suppose that the decrements can, in any circumstance whatever, bear to each other that proportion; but asserts, on the contrary, that the proportion of the fluxions is only a proportion limiting all the varying proportions that these decrements have to each other in their various degrees of diminution.

At the same time that this objection was rais'd against the doctrine of fluxions, the method of prime and ultimate proportions was also excepted to: in particular it was urg'd that the quantities or ratios asserted in this method to be ultimately equal, were frequently such as could never absolutely coincide. As for instance, the parallelograms inscrib'd within the curve in the second *Lemma* of the first book of Sir *Isaac Newton*'s *Principia*, cannot by any division be made equal to the curve they inscrib'd; whereas in that *Lemma* it is asserted that they are ultimately equal to it.

And here again two different methods of explanation have been given. The first, supposing that by ultimate equality a real assignable coincidence is intended, asserts, that these parallelograms and the curve do become actually, perfectly, and absolutely equal to each other. But the author of the above-mentioned treatise has given such an interpretation of this method, as did no ways require any such coincidence.

In his explication of this doctrine of prime and ultimate ratios, he defines the ultimate magnitude of any varying quantity to be the limit of that varying quantity which it can approach within any degree of nearness, and yet can never pass. And in like manner the ultimate ratio of any varying ratio is the limit of that varying ratio. These definitions being premis'd, he demonstrates that when varying magnitudes keep always in the same proportion, then their ultimate magnitudes will be in that same proportion, and that all the ultimate ratios of any particular varying ratio is the same. From these proportions thus establish'd, all that has at any time been demonstrated by the ancient method of exhaustions may be most easily and elegantly deduced; and that by a method not yielding in brevity to the artless inconclusive process by indivisibles.

It is evident that no coincidence of the varying quantity and its limit is at all suppos'd necessary in this method, since the ultimate magnitude of a varying one is not so denominated from any such coincidence of the varying one with it, but from its being that magnitude, which the varying one can approach within any degree of nearness.

It has indeed been suppos'd, that the accuracy of the demonstrations founded on this doctrine did in reality depend on this coincidence; but this mistake has arisen from forgetting that the demonstrations deduced from this method, are applied to the limits of varying magnitudes and proportions, and not to the varying quantities or proportions themselves.

Thus, if by means of a polygon describ'd about a circle, we were to demonstrate the equality of that circle to a triangle, having for its base the circumference of that circle, and the semidiameter for its altitude, the proof would not be founded on the real coincidence of the polygon and circle, since this could not be effected by any diminution of the Sides of the polygon; but the demonstration would altogether proceed by shewing, that the circumscrib'd polygon could approach both the circle and triangle within any degree of nearness, and yet could pass neither of them; therefore the circle and triangle thus shewn to be the limits or ultimate magnitudes of the same varying magnitude, cannot differ from each other.

In like manner in demonstrating the proportion that the fluxions of two flowing quantities bear to each other, the demonstration is not founded on the coincidence of the proportion of the decrements with that proportion which is given for the proportion of the fluxions; for the coincidence of these proportions cannot by any diminution of the decrements be ever affected: but the proof depends upon this, that since by diminishing the decrements, the proportion of those decrements can be brought within any degree of nearness to that given proportion, and also to the proportion of the fluxions, and yet can never pass either of them; therefore that given proportion, and the proportion of the fluxions, cannot differ from each other, they being thus shewn to be each of them the limit or ultimate proportion of the same varying proportion.

From hence it appears, that the coincidence of the variable quantity and its limit, could it be always prov'd, would yet bring no addition to the accuracy of these demonstrations; and since by the division of magnitudes no such coincidence can ever take place, why to the natural difficulty of these subjects should the obscurity of so strained a conception be added? Certainly neither brevity, perspicuity, nor exactness, can be at all promoted, by supposing in these demonstrations that circumstance to be ever necessary, which in numberless instances is not possible, and which by its taking place or not, can no ways affect the justness of the conclusion.

But it has been urg'd against this explication, that Sir *Isaac Newton* does in the first *Lemma* of his first book assert such a coincidence; and therefore though the method of managing prime and ultimate proportions here described may be conclusive, yet it is not a true interpretation of Sir *Isaac Newton*.

What foundation there is for this charge, will best appear by considering this *Lemma*; and that this may be done with more convenience, we will insert a literal translation of it.

Quantities, and the ratios of quantities, that during any finite time constantly approach each other, and before the end of that time approach nearer than any given difference, are ultimately equal.

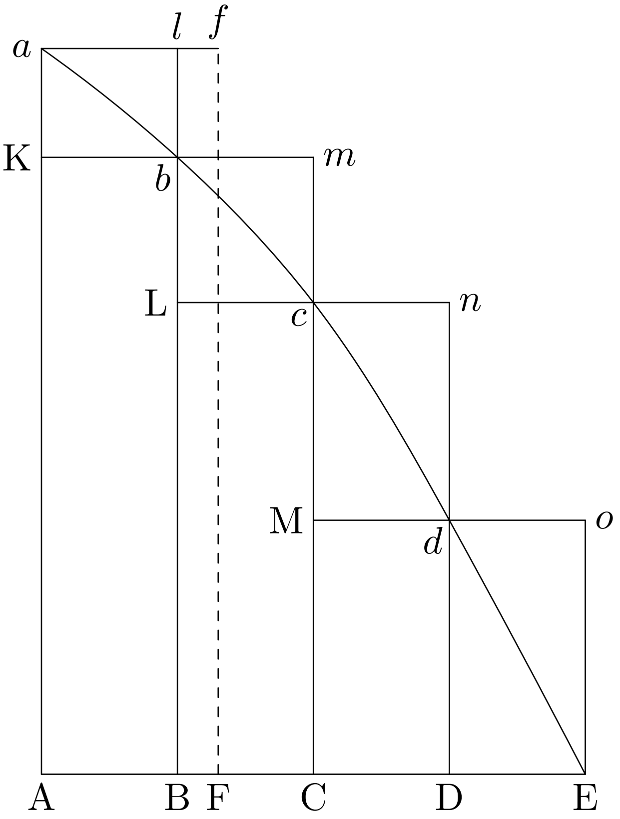
In order that the coincidence between the variable quantity and its limit should be intended in this *Lemma*, it is necessary that the phrase of *given difference* should mean a difference that may be taken at pleasure, after the celerity or degree of approach of these quantities or ratios is in every part determin'd.

But if, according to the most usual and authentic signification of this phrase, there is meant by the *given difference*, in this *Lemma*, a difference first assign'd, according to which the degree of approach of these quantities may be afterwards regulated; then variable quantities or ratios, and their limits, tho' they do never actually coincide, will come within the description of this *Lemma*; since the difference being once assign'd, the approach of these quantities may be so accelerated, that in less than any given time the variable quantity, and its limit, shall differ by less than the assign'd difference.

Now that the latter sense is the true interpretation, will appear from the demonstration and application of this *Lemma*.

In the first place, the demonstration of this *Lemma* may, without the change of a single word, be applied to prove that the ultimate ratio of the ordinate of the hyperbola to the same ordinate continued to the asymptote, is the ratio of equality; and yet it is confess'd, that in this case there can never be an actual coincidence.

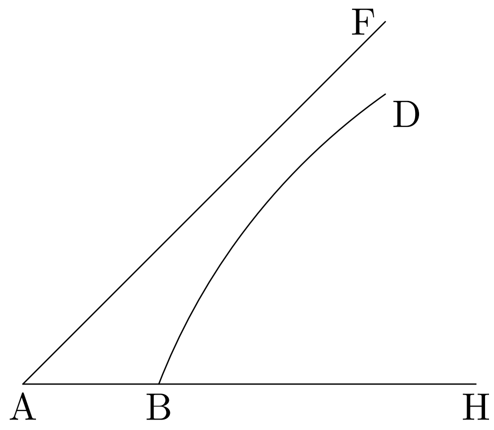
In the next place, the quantities in many of the succeeding *Lemmas*, to which the first is applied, are such where the approach is determin'd by a subdivision into parts; but by this method of proceeding it is obvious, that no coincidence can ever be obtain'd.



However, it is said that by motion this coincidence may be actually made to take place even in these quantities; as, suppose in the second *Lemma* a point E to describe the line E A with a continued motion in the space of an hour; and let it be conceiv'd, that in every point of time during that hour, a rectangle, as A B l, is raised upon A B, that part of the line E A which at the point of time is yet undecrib'd; also upon any other part of the line equal to A B let other rectangles be erected, as in the figure, at the same point of time. It is said that by this means, at the end of the hour, when the point E arrives at A, the curve and the inscrib'd figure will actually coincide.

To this it may be reply'd, that supposing the coincidence could by this means take place, it would prove that no such coincidence was ever intended by Sir *Isaac Newton*; since had he regarded it as a necessary circumstance, he would certainly have applied to this *Lemma* a method of inscribing the figure, by which such a coincidence might be shewn; whereas by describing the parallelograms by a continual division, and making their bases constantly equal, and always some *aliquot* part of the whole, he has necessarily excluded the description of them by motion, by which means only it is supposed that this coincidence can be brought about.

But further, this supposed demonstration, that an actual coincidence of the inscrib'd figure may be effected by the forementioned motion, is really inconclusive; since from a like method of proceeding may be deduced this absurd conclusion, that hyperbolas coincide with their asymptotes.



Suppose an hyperbola B D, its diameter A H, and its asymptote A F. Now let the line A H revolve about the center A with an angular motion, till it coincides with the line A F; then, since it is demonstrated by geometers, that every line drawn within the angle F A H will, if produc'd, meet with the hyperbola; it is evident that the line A H will meet with it in every part of its motion through the angle F A H. Moreover, as the line approaches the asymptote, the intersection thereof with the curve will continually become less and less distant from the asymptote; insomuch that this line may be made to approach so near the asymptote, that this intersection shall be less remote from it, than by any distance, how minute soever, that can be named. Now let us suppose the line A H to employ any given space of time, as an hour, in passing over the angle H A F; then does the intersection of the revolving line with the hyperbola continually approach to the asymptote during the space of this hour, and before the end of the hour this point in the hyperbola will approach nearer to the asymptote than by any difference that can be proposed; consequently by the method of reasoning above made use of, we must conclude, that at the end of the hour the hyperbola actually coincides with the asymptote.

If it be examin'd wherein lies the fallacy of these conclusions, it will be found, that tho' the meeting of the hyperbola and its asymptote, and the coincidence of the inscrib'd figure and the curve, seem to be pointed out and determin'd by this form of reasoning; yet to continued the hyperbola and asymptote till they actually meet, requires the delineation of a line longer than any line that can be assign'd; and to describe a figure within the curve under the suppos'd circumstance of coincidence, requires the delineation of a line less than any line that can be assign'd: both which operations are equally impossible.

It may perhaps be worth while to examine, how it happens that the meeting of the hyperbola and its asymptote should be acknowledged impossible, and yet the coincidence of the inscrib'd figure and curve so strongly contended for, when they each of them require a construction equally inconceivable and unattainable. The reason, I suppose, for this extraordinary partiality is, that as a quantity in augmenting without limit did most obviously pass beyond the utmost stretch of imagination, it was without difficulty granted, that the delineation or conception of any such magnitude was impossible. Whereas when a quantity diminish'd without limit, the imagination could trace it during the whole time of its diminution; and consequently the conception of a quantity less than any whatever, has been thought possible by some, who allow the absurdity of pretending to conceive a quantity greater than any whatever.

If it be said, that tho' the hyperbola and its asymptote cannot be describ'd under the circumstance of meeting each other; yet the inscrib'd figure and the curve can be describ'd under the circumstance of coincidence; for that the curve itself is the last form of the inscrib'd figure.

I answer, this is not true; for the supposed last form of the inscrib'd figure must essentially differ from the curve, the perimeter of the inscrib'd figure contiguous to the curve being in every description, and consequently in this imagin'd last equal to the sum of the lines aA, A E: whereas if the curve was really the last form of the inscribed figure, their perimeters could not differ. Since then the curve is not the last form of the inscrib'd figure, and since the last form of this figure cannot be describ'd, but by the delineation or conception of a line less than any line that can be assign'd; it is evident that the coincidence in this case does equally, with the meeting of the hyperbola and its asymptote, involve an impossibility.

But the strongest proof that Sir *I. Newton* does not always consider this coincidence of the variable quantity, or ratio and its ultimate as necessary in his method, is, that he himself tells us, that if two lines increasing without limit have always a given difference, then their ultimate ratio will be the ratio of equality. Now the phrase of ultimate ratio is peculiar to him and his method, and cannot possibly be suppos'd in this place to have a signification different from what it had in the first and subsequent *Lemmas*; consequently the ultimate ratio is, by his own express description, a ratio that the variable one it is ascrib'd to cannot always coincide with.