AN ACCOUNT OF A DISCOURSE ON THE NATURE AND CERTAINTY OF SIR ISAAC NEWTON'S FLUXIONS, AND OF PRIME AND ULTIMATE RATIOS

By

Benjamin Robins

(The Present State of the Republick of Letters, October 1735, pp. 245–270)

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NOTE ON THE TEXT

This account appeared in *The Present State of the Republick of Letters* for October 1735. It was reprinted (with some alterations) in *Mathematical Tracts of the late Benjamin Robins Esq.*, edited by James Wilson, London, 1761.

Two corrections have been made, in accordance with the sense of the passages in question, and in agreement with corrections made in the *Mathematical Tracts of the late Benjamin Robins Esq.*:

In the 11th paragraph, 5th line, F O has been corrected to G O;

In the 14th paragraph, 3rd line, F G has been corrected to F O.

The following spellings, differing from modern British English, are employed in the original 1735 text: surprizing, expresly, strait, compleat.

David R. Wilkins

Dublin, June 2002

A Discourse *concerning the Nature and Certainty of Sir* Isaac Newton*'s Methods of Fluxions, and of prime and ultimate Ratios. By* Benj. Robins*, F.R.S.* London: *Printed for* W. Innys *and* R. Manby*, at the West End of St.* Paul*'s Churchyard.* 1735. *In* 8vo. *Pag.* 78.

[*The Present State of the Republick of Letters*, October 1735, pp. 245–270.]

Some doubts having lately arisen concerning Sir *Isaac Newton*'s doctrines of fluxions, and of prime and ultimate ratios; this treatise was written with design to give such an idea of both these subjects, as might clear them from uncertainty, without entering into the discussion of any particular objections.

For this end the author has been careful, not only to distinguish both these methods from that usually known by the name of indivisibles, but also from each other.

The manner wherein the ancients demonstrated, what relates to the mensuration of curvilinear spaces, not giving any distinct notion of the principles, upon which they built their analysis of such problems; about thirty years before Sir *Isaac Newton* invented his method of fluxions, *Cavalerius*, a mathematician of *Italy*, proposed in these problems a new form of reasoning*. He supposes, that all surfaces might be filled up with parallel lines, and all solids by parallel planes; and then lays down this fundamental proposition, That different planes are in the same proportion to each other as all the lines contained in each, and different solids in the same proportion as all the planes contained in them. As this is a manner of expression hardly accompanied with any ideas; since it is not at all intelligible to speak of the collective number of lines or planes, where their number is wholly undetermined and infinite: this method was from the beginning opposed, as ungeometrical, and in no measure agreeable to that clearness of conception and expression, for which the mathematical sciences had ever been celebrated. But however, as by proper cautions error in the conclusions might be avoided, and this method promised great assistance in the analysis of a subject, wherein the ancients had made the least progress; it was warmly espoused by *Torricellius*, and others of the most illustrious geometers. Our countryman Dr. *Wallis*, and many after him, thought it an improvement of this method to substitute, in the room of lines, parallelograms, whose breadth was to be called infinitely small; and for the planes, whereby solids were supposed to be filled up, prisms or cylinders of an infinitely small altitude: so that the method of indivisibles at length supposed all geometrical magnitudes, whether lines, surfaces, or solids, to be composed of an infinite number of homogeneous magnitudes, each infinitely small.

But this is a mode of expression no way more intelligible than the other. Sir *Isaac Newton* therefore instituted a manner of conception upon quite different principles. He observing (to use his own words) that indivisibles have no being either in geometry or in nature†; instead of this infinite and inconceivable subdivision of magnitudes already formed, he considered

* In Geometria Indivisibilibus promota, *Edit.* 1635.

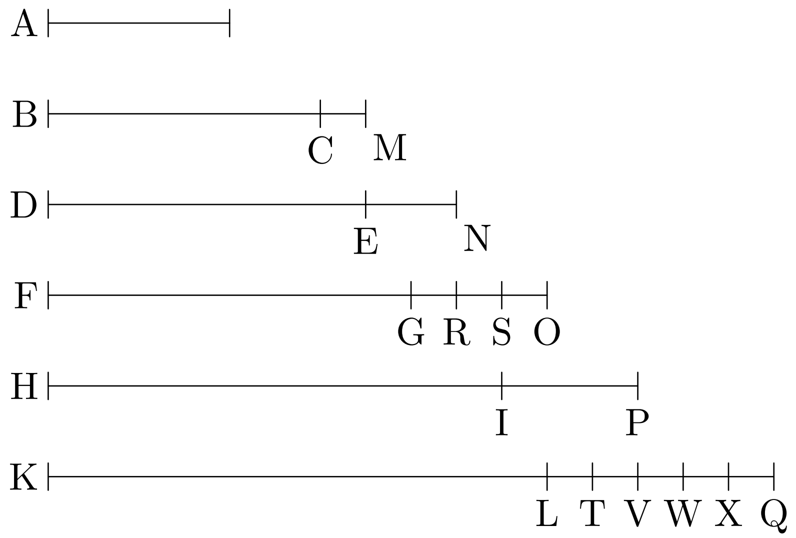
† Philos. Transact. N$^{\text{o}}$ 342. pag. 205.

them as produced before the imagination by some motion. And thus the same magnitude will in some parts increase faster than in others, and different magnitudes described together increase by different degrees of swiftness. Now if the proportion between the celerity of increase of two magnitudes produced together is in all parts known; it is evident, that the relation between the magnitudes themselves must from thence be discoverable.

This is the foundation of his method of fluxions. And to form a true idea thereof, we must distinguish between the increase, which a line or figure receives in any given space of time, and the velocity, wherewith that increase is generated: for though by the velocity, wherewith any line or figure continually augments, the quantity of its increase may be known; and on the contrary, from the quantity of the increase the velocity, wherewith that increase was produced; yet the quantity thus added is not the velocity, wherewith it is generated. And the method of fluxions requires the knowledge of these velocities of increase only: as Sir *Isaac Newton*'s other method of prime and ultimate ratios proceeds entirely upon the consideration of the increments produced.

In this method are introduced no forms of expression, but what convey very clear and distinct ideas, and such as have not the least affinity with the mode of conception schemed out in that of indivisibles. This manner therefore of considering magnitudes, as they are gradually produced before the mind, is a genuine method of discovering the relation between such magnitudes, which may afterwards be proved to bear that relation by a subdivision into parts, as practiced by the ancient geometers; since it is shewn in this discourse, that the rules for finding fluxions are demonstrable according to the ancient forms.

The demonstration here given for the fluxion of a power, is formed upon the model of that delivered by Sir *Isaac Newton* himself; but may be otherwise performed after the following manner.



Let A, B C, D E, F G, H I, K L, and A, B M, D N, F O, H P, K Q be two series's of continued proportionals beginning with the same term A; then if h be a number denoting the distance of the terms K L, K Q from A, and k a number denoting the distance of the terms F G, F O from the same; the ratio of L Q, the difference of the terms K L, K Q the most remote from A, to G O, the difference of the other terms F G, F O, is greater than the ratio of $h \times$ K L to $k \times$ F G, and less than the ratio of $h \times$ K Q to $k \times$ F O.

The ratio of D E to D N is the duplicate of the ratio of B C to B M; for the ratio of D E to A is the duplicate of that of B C to A, and the ratio of A to D N is the duplicate of that of A to B M; therefore, by equality, the ratio of D E to D N is the duplicate of the ratio of B C to B M.

In like manner the ratio of F G to F O is the triplicate of the ratio of B C to B M, and so of the rest. Insomuch that between any two terms in these series's, equally distant from the first term A, as many mean proportionals in the ratio of B C to B M will fall, as are the number of the intermediate terms in either series between these terms and A. Thus between F G and F O fall the mean proportionals F R, F S, and between K L, K Q, the mean proportionals K T, K V, K W, K X. Here the differences G R, R S, S O will be equal in number to k, by which is denoted the distance of the term F G from A, and the differences L T, T V, V W, W X, X Q will be equal in number to h. Now K L is to F G as L T to G R; and L T, T V, V W, W X, X Q are in the same continued proportion as G R, R S, S O: therefore K L is to F G as L W, the sum of L T, T V, V W, whose number is k, to G O, the sum of G R, R S, S O, whose number is likewise k. But the ratio of L Q to G O is compounded of the ratio of L Q to L W, and of that of L W to G O. Now the ratio of L Q to L W is greater than the ratio of h to k; therefore the ratio of L W to G O being the same of that of K L to F G, the ratio of L Q to G O is greater than that compounded of the ratio of h to k, and of that of K L to F G; that is, greater than the ratio of $h \times$ K L to $k \times$ F G.

Again: K Q is to F O as Q X to O S; therefore Q X, X W, W V, V T, T L being in the same continued proportion with O S, S R, R G; K Q is to F O as Q V, the sum of Q X, X W, W V whose number is k, to G O. But the ratio of L Q to G O is compounded of the ratio of L Q to Q V, which is less than the ratio of h to k, and of the ratio of Q V to G O, or of the ratio of K Q to F O. Therefore the ratio of L Q to G O is less than that of $h \times$ K Q to $k \times$ F O.

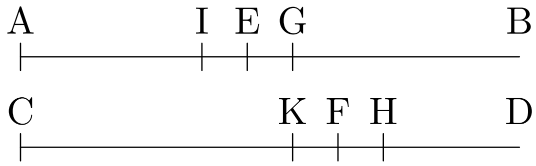
Again, I say, that L Q and G O may be taken so small, that the ratio of L Q to G O shall be less than any ratio, that shall be proposed greater than the ratio of $h \times$ K L to $k \times$ F G; or greater than any ratio, that shall be proposed less than the ratio of $h \times$ K Q to $k \times$ F O.

Let D E, D N be the terms in these series's at the same distance from A, as K L is from F G; then D E will be to A as K L to F G, and $h \times$ D E to $k \times$ A as $h \times$ K L to $k \times$ F G; likewise D N to A as K Q to F O, and $h \times$ D N to $k \times$ A as $h \times$ K Q to $k \times$ F O.

Now in the first place take $h \times$ D N in a less ratio to $k \times$ A, than the ratio proposed greater than that of $h \times$ K L to $k \times$ F G. Then the ratio of $h \times$ K Q to $k \times$ FO will be less than the ratio proposed; but the ratio of L Q to G O is less than that of $h \times$ K Q to $k \times$ F O, and therefore will be less than the ratio proposed.

In the next place, let $h \times$ D E be taken to $k \times$ A in a greater ratio than than the ratio proposed less than that of $h \times$ K Q to $k \times$ F O. Then will the ratio of $h \times$ K L to $k \times$ F G be greater than that now proposed. But the ratio of L Q to G O is greater than that of $h \times$ K L to $k \times$ F G, consequently greater than that now proposed.

Now in the figure at page 7. of this book, A E being denoted by x, let C F be denoted by $\frac{x^{\frac{m}{n}}}{a^{\frac{m-n}{n}}}$, m and n representing any two whole numbers. Then AE and C F will be two terms in a series of proportionals beginning from a, the number n denoting the place of A E, and m the place of C F. Here I say, the velocity, wherewith the point describing C D moves at F, is



to the velocity of the point moving on A B at E, as $m \times$ C F to $n \times$ A E; that is as $m \times \frac{x^{\frac{m}{n}}}{a^{\frac{m-n}{n}}}$ to $n \times x$, or as $\frac{m}{n} x^{\frac{m-n}{n}}$ to $a^{\frac{m-n}{n}}$.

In the first place G and H, as likewise I and K being other contemporary positions of the points moving on A B and C D; A G and C H will be terms in a series of proportionals beginning from a, and likewise A I and C K terms in another series of proportionals also beginning from a situated in like manner, as the terms A E and C F in the series, to which they belong. Therefore, if m be greater than n, from what has above been written, F H bears a greater proportion to E G, and K F a less proportion to E I than $m \times$ C F bears to $n \times$ A E. Consequently, if I E be equal to E G, K F will be less than F H; insomuch that if the point on A B moves with a uniform velocity, the point on C D moves with a velocity continually accelerated.

Now, if possible, let the velocity at F bear to the velocity at E a greater proportion than that assigned, suppose the ratio of p to q.

Because the ratio of p to q is greater than that of $m \times$ C F to $n \times$ A E, let the ratio of $m \times$ C H to $n \times$ A E be less than the ratio of p to q. Then, by what has been above written, the ratio of F H to E G is less than the ratio of $m \times$ C H to $n \times$ A E, consequently less than the ratio of p to q, or of the velocity at F to the velocity at E; which is absurd, the first of these ratios being greater than the last.

Again, suppose the velocity at F bear to the velocity at E a less proportion than that assigned, suppose the ratio of r to s.

The ratio of r to s being less than that of $m \times$ C F to $n \times$ A E, let the ratio of $m \times$ C K to $n \times$ A E be greater than the ratio of r to s. Then the ratio of K F to I E will be greater than the ratio of $m \times$ C K to $n \times$ A E; consequently greater than the ratio of r to s, or of the velocity at F to the velocity at E; which is absurd, the first of these ratios being less than the last.

If the point on the line A B should move from I to G, not with a uniform velocity, but with a velocity continually increasing; since, when I E is equal to E G, K F is less than F H, the point on C D will move with a velocity more accelerated; and if the point moving from I to G proceed with a decreasing velocity, the motion from K to H will at least decrease slower: insomuch that in these cases also the proportion of F H to E G will be greater, and that of K F to I E less than that of the velocity at F to the velocity at E. Therefore the demonstration will here proceed in the same manner as before.

If m be less than n, K F will be greater than F H, and the point on the line C D move with a decreasing velocity, when the motion from I to G is uniform; but the demonstration will here also proceed after the like manner. Nor will it be different when one of the numbers m or n is negative.

The doctrine of fluxions, as delivered by Sir *Isaac Newton*, consists of two parts; the form of conception we have above described, and the method of applying it for the solution of mathematical problems. The surprizing improvements Sir *Isaac Newton* has made in the analytical part of geometry by these principles, his immortal treatise on the quadrature of curves abundantly sets forth. But the author of this discourse designing only to consider the mode of conception proposed in this doctrine, he has avoided any particular explanation of the forms of calculation, that having been largely performed by others. But as the introduction to this subject, the most read, has accommodated these calculations to the system of indivisibles*, it has occasioned the true mode of conception, so necessary to give this doctrine a place in geometry, to be very much neglected. The author therefore has shewn at large, how we may carry along with us the genuine form of conception in the application of this doctrine to the mensuration of curvilinear spaces, the drawing of tangents, and other problems, to which these principles are to be applied.

Fluxions not affording the most convenient means for synthetic demonstration, Sir *Isaac Newton*, who in all his writings has shewn the strongest desire of using brevity, invented still another form of reasoning, from what he calls the prime and ultimate ratios of the increments or decrements of varying quantities, whereby to avoid the length of the ancient demonstrations by exhaustions.

This method is as essentially different from that of indivisibles, as the former; but however, requires somewhat greater attention to avoid falling into that faulty manner of conception. It depends on the first lemma of his mathematical principles of natural philosophy, the genuine meaning of which is, That those quantities are to be esteemed ultimately equal, and those ratios ultimately the same, which are perpetually approaching each other in such a manner, that any difference how minute soever being given, a finite time may be assigned, before the end of which the difference of those quantities or ratios shall become less than that given difference.

What Sir *Isaac Newton* intends we should understand by the ultimate equality of magnitudes, and the ultimate identity of ratios proposed in this lemma, will be best known from the demonstration annexed to it. By that it appears, Sir *Isaac Newton* did not mean, that any point of time was assignable, wherein these varying magnitudes would become actually equal, or the ratios really the same; but only that no difference whatever could be named, which they should not pass. The ordinate of any diameter of an hyperbola is always less than the same continued to the asymptote; yet the demonstration of this lemma can be applied, without changing a single word, to prove their ultimate equality. The same is evident from the lemma immediately following, where parallelograms are inscribed, and others circumscribed to a curvilinear space. Here the first lemma is applied to prove, that by multiplying the number and diminishing the breadth of these parallelograms *in infinitum*, that is, perpetually and without end, the inscribed and circumscribed figures become ultimately equal to the curvilinear space, and to each other; whereas it is evident, that no point of time can be assigned, wherein they are actually equal; to suppose this were to assert, that the variation ascribed to these figures, though endless, could be brought to a period, and be perfectly accomplished; and thus we should return to the unintelligible language of indivisibles. The excellence of this method consists in making the same advantage of this endless approximation towards equality, as by the use of indivisibles, without being involved in the absurdities of that doctrine. In short, the difference between these two may be thus explained.

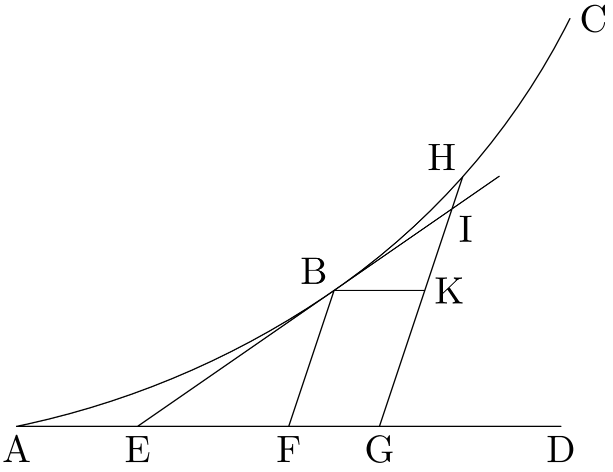
* Analyse des infiniment petits. Par le Marquis de l'*Hôpital*.

There are but three ways in nature of comparing spaces: one is by shewing them to consist of such, as by imposition on each other will appear to occupy the same place: another is by shewing their proportion to some third; and this method can only be directly applied to the like spaces as the former, for this proportion must be finally determined by shewing, when the multiples of such spaces are equal, and when they differ: the third method, to be used where these other two fail, is by describing upon the spaces in question such figures, as may be compared by the former methods, and thence deducing the relation between those spaces by that indirect manner of proof commonly called *deductio ad absurdum*; and this is as conclusive a demonstration, as any other, it being indubitable, that those things are equal, which have no difference. Thus *Euclide* and *Archimedes* demonstrate all they have writ concerning the comparison and mensuration of curvilinear spaces. The method advanced by Sir *Isaac Newton* for the same purposes differs from theirs, only by applying this indirect form of proof to some general propositions, and from thence deducing the rest by a direct form of reasoning. Whoever compares the fourth of Sir *Isaac Newton*'s lemmas with the first, will see, that the proof of the curvilinear spaces there considered having the proportion named depends wholly upon this, that if otherwise the figure inscribed within one of them could not approach by some certain distance to the magnitude of that space: and this is precisely the form of reasoning, whereby *Euclide* proves the proportion between different circles. As this method of reasoning is very diffusely set out in the writings of the ancients, and Sir *Isaac Newton* has here expressed himself with that brevity, that the turn of his argument may possibly escape the unwary; the author has recommended the reading the ancients, as the best introduction to the knowledge of this method. The impossible attempt of comparing curvilinear spaces without having recourse to the forementioned indirect method of arguing produced the absurdity of indivisibles.

As the magnitudes called in this lemma ultimately equal may never absolutely exist under that equality; so the varying magnitudes holding to each other the variable ratios here considered may never exist under that, which is here called their ultimate ratio. Of this Sir *Isaac Newton* gives an instance, which the author of this treatise has repeated after him, from lines increasing together by equal additions, and having from the first a given difference. For the ultimate ratio of these lines in the sense of this lemma, as Sir *Isaac Newton* himself observes, will be the ratio of equality, though these lines can never have this ratio; since no point of time can be assigned, when one does not exceed the other.

In like manner the quantities called by Sir *Isaac Newton* vanishing may never subsist under that proportion here esteemed their ultimate.

In page 64 of this treatise, where B F bears the same proportion to the subtangent F E, as that wherewith the lines H K, K B vanish, these lines must not be conceived, by the name of evanescent or any other appellation, ever to subsist under that proportion; for should we conceive these lines in any manner to subsist under this proportion, though at the instant of their vanishing, we shall fall into the unintelligible notion of indivisibles, by endeavouring to represent to the imagination some inconceivable kind of existence of these lines between their having a real magnitude, and becoming absolutely nothing. Sir *Isaac Newton* was himself apprehensive, that this mistake might be made; for as he thought fit (in compliance with the bad taste, which then prevailed) to continue the use of some loose and indistinct expressions resembling those of indivisibles, for which he has himself apologised, he expresly cautions us against misinterpreting him in this manner, when he says: *Si quando dixero quantitates quam*



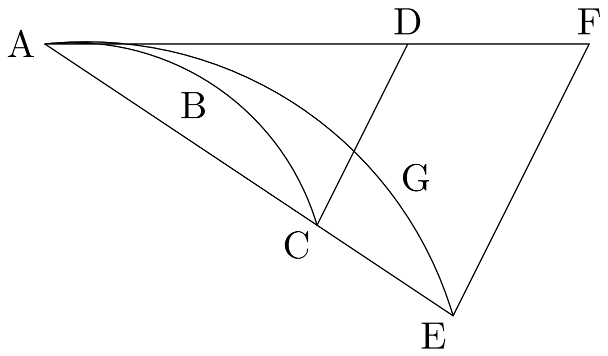
minimas, vel evanescentes, vel ultimas, cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite. Thus expresly has he declared to us, that vanishing quantities, or whatever other less accurate appellation he names them by, are to be considered as indeterminate quantities bearing to each other under their different magnitudes different proportions; and that the limit of these proportions, which the quantities themselves can never obtain, is that, for the sake of which, these quantities are considered: insomuch, that since these quantities have different proportions, while they obtain the name of vanishing quantities, the author of this treatise has justly observed the term ultimate to be necessarily added to denote that proportion, which is the limit of an endless number of varying ones. The like remark is necessary, when these quantities are considered in the other light as arising before the imagination: for then the proportion intended must be specified by calling it the first or prime proportion of these quantities. And as this additional epithet is necessary to express the proportion intended, so it is absurd to apply it to the quantities themselves; as Sir *Isaac Newton* says, there are *rationes primæ quantitatum nascentium*, but not *quantitates primæ nascentes**.

The author of this treatise thought the readiest method to guard against all errors of this kind, was to represent the principles of prime and ultimate ratios, and their application to geometrical subjects under such a form of expression, as might be so totally inconsistent with indivisibles, as not to be capable by any misinterpretation of being accommodated to that erroneous manner of conception. But at the same time he took care, that his phrase should not differ essentially from Sir *Isaac Newton*'s: as will appear by comparing the two modes of expression in the following instance, being the seventh of Sir *Isaac Newton*'s lemmas concerning prime and ultimate ratios, which is to this effect.

If any arch A B C be subtended by a chord A C, and at A, where the curvature is understood to be uninterrupted, it be touched by the strait line A D; if the point C be supposed to approach towards A, till those points coincide, the ultimate ratio of the chord, arch, and tangent will be the ratio of equality; provided the tangent A D be terminated by some line D C drawn from C, the extremity of the arch, so as always to make some angle with the tangent A D.

This Sir *Isaac Newton* thus demonstrates.

* Philosophical Transact. N° 342. pag. 205.



While the point C approaches to the point A, suppose A C and A D always to be produced to the distant points E and F, and E F to be drawn parallel to C D; and let the Arch A G E be always similar to the arch A B C. Now when the points A and C coalesce, the rectilinear angle under E A F must vanish; therefore the right lines A E, A F, which are always of a finite magnitude, and also the intermediate arch A G E must coincide, and consequently become equal. Therefore the ultimate ratios of the strait lines A C, A D, and the curve A B C, all which vanish, when the point C coincides with A, will be the ratio of equality.

Now in the phrase of this book, it must be said, that the arch A G E can never be equal to the chord A E; nor the chord A E equal to the tangent A F, unless when the angles under A E F and under A F E chance to be equal. But, by causing the point C to approach the point A, the ratios of these three lines to each other may at last be brought nearer to the ratio of equality than to any other whatever. Therefore, according to the definition in pag. 57. of this treatise, the ultimate ratio of any one of these three lines to either of the others is the ratio of equality. And again, since the straight lines A C, A D, and the arch A B C are always to each other in the same proportion with the lines A E, A F, and the arch A G E; the ultimate ratios of the three lines A C, A D, A B C will be the same with the ultimate ratios between A E, A F, A G E, by the proposition subjoined to that definition: therefore the ultimate ratio between any two of these lines A C, A D, A B C is the ratio of equality.

By this instance it is manifest, that the style, under which the author has treated this subject, is only an interpretation of Sir *Isaac Newton*'s; and such an interpretation he thought alone sufficient to answer the purpose of his writing. No objection had been made against the truth of the conclusions drawn from this method of reasoning. Indeed all error of that kind may be avoided by proper circumspection even in the use of indivisibles. But as the only suspicion lay against the propriety in the conception and expression advanced in this doctrine; if his interpretation involves in it no perplex'd or imperfect ideas, which the author flatters himself will be allowed, it is a full justification of this method.

The author believes himself as desirous, as any one can be, to preserve propriety of expression and perspicuity of conception in mathematical subjects. He therefore freely acknowledges, that he has not vindicated this doctrine, unless he shall be found to have accommodated to it a clear and unexceptionable mode of expression, nor freed the inventor from censure, unless he has also shewn, that the turn of phrase recommended by Sir *Is. Newton*, without any forcible construction, is adapted to convey the same ideas. The author has often lamented the negligence of geometrical writers in regard to their style and diction. The introduction of the terms of arithmetic into geometry by *Des Cartes*, and the favourable reception of the unintelligible jargon of indivisibles have overwhelmed the mathematical sciences with such a profusion of intricate and inconceivable forms of speaking, that they began to be no longer that guide to sound reasoning, which they had hitherto been thought. To restore in some measure geometry from this corruption was the design of Sir *Isaac Newton* in advancing the doctrine of prime and ultimate ratios; and how far the author of this treatise has proved him successful, must be submitted to the judgment of the publick.

Sir *Isaac Newton* has made use of prime and ultimate ratios chiefly for synthetic demonstration; yet as they furnish a direct manner of proof, it is manifest they may be also applied to the analysis of problems.

To compleat the design of this treatise, it was necessary to explain Sir *Isaac Newton*'s own demonstrations of his rules for finding fluxions; for which purpose nothing more was thought necessary than to dilate Sir *Isaac Newton*'s words by a small paraphrase. For though Sir *Isaac Newton* demonstrates these rules upon the principles of prime and ultimate ratios, yet after what had been written, it seemed scarce possible any longer to confound these two methods together. Indeed when the author considers, how expresly Sir *Isaac Newton* himself has distinguished them, he owns himself surprised, that this mistake should ever have been made.

The treatise concludes with an explication of what is to be understood by the momentum of quantities, which is a term appertaining to the doctrine of prime and ultimate ratios only*.

The term momentum being of no other use than to give the expression in particular cases greater brevity, the truth of this doctrine has no dependance on the sense of this term; therefore it was not necessary to be taken notice of in this general account; but as it has been conceived to contain something very abstruse, if not unintelligible, the author has explained it at large.

And here the author confesses he met with the greatest difficulty; for it must be acknowledged, that Sir *Isaac Newton*'s description is capable of an interpretation too much resembling the language of indivisibles. But were we to allow Sir *Isaac Newton*'s definition of momenta to be founded entirely upon that erroneous doctrine, the utmost that will follow from so large a concession is only this, that though he invented the doctrines of fluxions, and of prime and ultimate ratios; yet he has demonstrated some of the propositions in his mathematical principles of natural philosophy by the means of indivisibles. It cannot hence be inferred, that his doctrine of prime and ultimate ratios has any connexion with indivisibles, or was insufficient for these cases. The author of this treatise has certainly freed the doctrine from this latter imputation by shewing, that such a sense may be put upon the word momentum, as will render those very demonstrations of Sir *Isaac Newton*, where this word is used, as just as any other upon the principles of prime and ultimate ratios. For this purpose the author, without confining himself to the express words of Sir *Isaac Newton*, has given such a definition of this term, as he though most suitable to the principles of that method.

The author of this treatise undertook not to prove, that Sir *Isaac Newton* has never deviated from the utmost propriety of expression, nor that he never demonstrated any proposition

* These symbols o and $\dot{x}$ are put for things of a different kind: the one is a moment, the other a fluxion or velocity.

Mr. *Leibnitz* hath no symbols of fluxions in his method. He used the symbols of moments or differences dx, dy, dz. *Philos. Trans.* N. 342. p. 204, 205.

upon the principles of indivisibles. He knows Sir *Isaac Newton* did sometimes make use of that method of reasoning; but this he contends for, that the methods under consideration are absolutely different from that doctrine.

When Sir *Is. Newton* first invented his method of fluxions, he demonstrated the rules of that method by indivisibles, as he acknowledges himself*, That in his calculus he made use of the character or symbol o to denote an infinitely small quantity, both in his *Analysis per æquationes numero terminorum infinitas*, and also in his treatise on quadratures, meaning as he at first writ it, from which we have a transcript in Dr. *Wallis*†, not as he afterwards corrected it in his own edition of that treatise. He has likewise made use of the same in one proposition of his Principles of natural philosophy‖. And he informs us, that though in demonstrating any proposition he chose to use the letter o for a finite moment of time, *&c.* and perform the whole calculation by the geometry of the ancients in finite figures or schemes without any approximation, and when the calculation was at an end, and the equation reduced, to suppose the moment o to decrease *in infinitum* and vanish; yet in the investigation, when it would make dispatch, he would suppose the moment o to be infinitely little, using all manner of approximation, which he conceived would produce no error in the conclusion‡. Accordingly we find in his treatise of quadratures, he freed his demonstrations from this defect, under which they first laboured; and the proposition of his Principles of philosophy, where he continued the use of indivisibles, is only the analysis of a problem.

Thus it appears, that Sir *Isaac Newton* did sometimes allow himself the use of indivisibles; but it also appears, that he always had a dislike to that method, as we learn from his own words, when he says, Since we have no ideas of infinitely little quantities, he introduced fluxions, that he might proceed by finite quantities as much as possible§. But as the brevity, wherewith he chose to write, obliged him still to have recourse to indivisibles for demonstrating the rules of this new method, he at length invented his other method of prime and ultimate ratios, and thereby entirely got over that difficulty.

We may likewise hence learn, how it came to pass, that his definition of momenta should contain expressions bearing some analogy to those of indivisibles: for he informs us, that originally he used the word moment in a sense agreeable to that doctrine; telling us, That from the moments of time he gave the name of moments to the momentaneous increases, or infinitely small parts generated in moments of time¶; though in his principles of philosophy he directs us to interpret his meaning according to the doctrine of prime and ultimate ratios, where he says, *neque spectatur magnitudo momentorum sed prima nascentium proportio***; and the author of the present treatise has accommodated his description of momenta to this,

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* Philos. Trans. N° 342. pag. 205. In his (*Newton*'s) calculus there is but one infinitely little quantity represented by a symbol, the symbol o.

Mr. *Newton* used the letter o in his analysis, and in his book of quadratures, and in his *principia philosophiæ*, and still uses it in the very same sense as at first. *Ibid.* p. 204.

† Oper. Vol. II. p. 392.

‖ Lib. II. prop. 10.

‡ Philos. Trans. N° 342. p. 179.

§ Ibid. p. 205.

¶ Ibid. p. 178.

** Lib. II. Lem. 2.

which he conceives to be Sir *Isaac Newton*'s intention.

As the proportion between the increments of magnitudes is in this doctrine considered only for discovering, what is here called their ultimate ratio; when the real proportion of these increments is not to be expressed, but by terms too complex, it is convenient, by neglecting some superfluous part of the increments, or by a proper addition to them, to form new quantities, which shall not only bear to each other a more simple proportion, but the ultimate ratio also of each quantity, thus formed, to the increment, whence it is deduced, shall be the ratio of equality: for by the proportion of such quantities the ultimate ratios of the increments are more readily assignable. These are the quantities called momenta. For example; if the increment of any line denoted by x be represented by o, the increment of the line denoted by any power x^n will be

$$nx^{n-1}o + n \times \frac{n-1}{2}x^{n-2}oo + \&c.$$

Here as the ultimate ratio of the first of these increments to the last is that of o to $nx^{n-1}o$, the line denoted by this term $nx^{n-1}o$ only is sufficient to express that ultimate proportion, and therefore may be assumed for the momentum of x^n; and then, to preserve a similitude of phrase, the entire increment of x is also to be called the momentum of that line.

Although it is a great mistake to suppose the validity either of the doctrine of fluxions, or of that of prime and ultimate ratios, to depend upon what Sir *Isaac Newton* has demonstrated concerning the momenta of quantities; yet since his demonstration of the momentum of a rectange had been controverted, the author has given a brief account of the principles, upon which that demonstration proceeds. And this may be represented more at large as follows.

To give this demonstration its utmost extent, suppose some third variable line Z, to which A and B, the sides of the rectangle in question, are in any manner related; and let a and b not be the real increments of A and B, but bear to the increment of z the most simple relation, whereby they can express the ultimate ratio of the increments of A and B to the correspondent increment of Z; then may a and b be called the momenta of A and B respectively. And since, at the same magnitudes of Z, A, and B the ultimate ratios between their decrements are the same with those between their increments, by the same magnitudes a and b may be also expressed the ultimate ratios of the decrements of A and B to the correspondent decrement of Z.

In like manner the ultimate ratio of the increment of the rectangle under A, B to the correspondent increment of any other rectangle under A and some given line M will be the same with the ultimate ratio of the decrement of the rectangle $\mathrm{A} \times \mathrm{B}$ to the correspondent decrement of the rectangle $\mathrm{A} \times \mathrm{M}$ at the same magnitudes of A and B. Therefore the ultimate ratio of the increment of $\mathrm{A} \times \mathrm{B}$ to the correspondent increment of $\mathrm{A} \times \mathrm{M}$, will be the same with the sum of such increment and decrement of $\mathrm{A} \times \mathrm{B}$ to the sum of the correspondent increment and decrement of $\mathrm{A} \times \mathrm{M}$.

Farther, the ultimate ratio of the increment of $\mathrm{A} \times \mathrm{B}$ to the correspondent increment of $\mathrm{A} \times \mathrm{M}$ will be the same with the ultimate ratio of that augmentation, which the rectangle under A, B will receive by increasing the Sides, either by their respective momenta, or by analogous parts of those momenta, to the augmentation, which the rectangle under A, M will receive from the moment of A, or a similar part thereof. Therefore the ultimate ratio of the increment of $A \times B$ to the correspondent increment of $A \times M$ will be the same with the ultimate ratio of that augmentation, which the rectangle $A \times B$ will receive from increasing its sides A and B by half their momenta a and b, to the augmentation, which $A \times M$ will receive from increasing A by half its momentum a. In like manner the ultimate ratio of the decrement of $A \times B$ to the correspondent decrement of $A \times M$ will be the same with the ultimate ratio of that diminution, which the rectangle $A \times B$ will receive by taking from each of its sides half its momentum, to the diminution, which the rectangle $A \times M$ will receive from half the momentum of A.

Hence it follows, that the ultimate ratio of the increment of $A \times B$ to the correspondent increment of $A \times M$ is the same with the ultimate ratio of the sum of the augmentation and diminution, which the rectangle $A \times B$ will receive from half the momenta of its sides, to the sum of the augmentation and diminution, which the rectangle $A \times M$ will receive from half the momentum of A; that is, the same with the ultimate ratio of $\frac{1}{2}a \times B + \frac{1}{2}b \times A + \frac{1}{4}ab$ and $\frac{1}{2}a \times B + \frac{1}{2}b \times A - \frac{1}{4}ab$ together, or of $a \times B + b \times A$, to $a \times M$.

As the square is comprehended under this general proposition for all rectangles, so by a similar artifice we may demonstrate the momentum of any other power. For instance, in the cube of any variable quantity A, whose increment or moment is o, if we divide that increment into two parts p and q, that the ratio of p to q may be subduplicate of the ratio of $3A - q$ to $3A + p$; $3A \times pp + p^3$ will be equal to $3A \times qq - q^3$, whereby the cube of $A + p$, or $A^3 + 3A^2p + 3Ap^2p^3$, will exceed the cube of $A - q$, or $A^3 - 3A^2q + 3Aq^2 - q^3$, by $3A^2 \times \overline{p+q}$, or $3A^2 \times o$, the momentum of A^3.

Here dividing the increment o into two equal parts will not answer the purpose intended; for by deducting the cube of $A - \frac{1}{2}o$, or $A^3 - \frac{3}{2}A^2o + \frac{3}{4}Ao^2 - \frac{1}{8}o^3$, from the cube of $A + \frac{1}{2}o$, or $A^3 + \frac{3}{2}A^2o + \frac{3}{4}Ao^2 + \frac{1}{8}o^3$, the residue will be $3A^2o + \frac{1}{4}o^3$, exhibiting more than is necessary for the momentum of the cube of A; for the momentum should be the simplest term, whereby the intended ultimate ratio can be express'd.

In other compound quantities the demonstration may be conducted upon the same model by such a division, as each particular case shall require, of the moments of the original quantities, whereof those under consideration are compounded; but when such division is of too perplex a kind, another method of demonstrating is to be preferred.

This is abundantly sufficient for explaining the demonstration in question. And as the author of this discourse presumes he has given throughout a genuine representation of Sir *Isaac Newton*'s real and sole intention: so he hopes it will appear, that however less exact in the choice of his expressions that great man may have been at any other time; yet when he purposely describes these methods, and explains their principles, he is not only perfectly consistent with himself, but has also delivered his meaning with such perpicuity that we need not have recourse to any deference for his authority to be fully satisfied of the truth of these doctrines.