CONSIDERATIONS OCCASIONED BY A PAPER IN THE LAST REPUBLICK OF LETTERS CONCERNING SOME LATE OBJECTIONS AGAINST THE DOCTRINE OF FLUXIONS

By

James Jurin

(The Present State of the Republick of Letters, January 1736, pp. 72–91)

Edited by David R. Wilkins

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NOTE ON THE TEXT

This text is transcribed from *The present state of the republick of letters* for January 1736.

The following spellings, differing from modern British English, are employed in the original 1736 text: expresly, encreasing.

In the paragraph beginning 'Having now fixed the meaning of the word *ultimo* in the *Lemma*,...', the phrase '*fiant ultimo æquales*' in the original 1736 text has been corrected to read '*fiant ultimo inæquales*', in accordance both with Newton's text, in the 3rd edition of *Philosophiæ Naturalis Principia Mathematica*, here referred to, and the translation given by Jurin ('let them become unequal') immediately following.

David R. Wilkins
Dublin, June 2002

Considerations *occasioned by a Paper in the last* Republick of Letters*, concerning some late* Objections *against the Doctrine of* Fluxions*, and the different Methods that have been taken to obviate them. By* Philalethes Cantabrigiensis.

[*The Present State of the Republick of Letters*, January 1736, pp. 72–91.]

Soon after the publication of my *second Letter* to the author of the *Analyst*, in defence of Sir *Isaac Newton*'s doctrine of Fluxions, came out a *Discourse* upon the same subject, written by a very ingenious and learned Gentleman; who, without entering into any controversy, has nevertheless treated of the points in dispute in such a manner, as if carefully attended to, might effectually silence all the objections that have been made against the truth and certainty of this doctrine. Particularly, observing the demonstration of Sir *Isaac*'s fundamental rule, for finding the proportion between the Fluxions of different flowing quantities, to be called in question, and specious exceptions raised against it; he has given us a new and masterly demonstration of his own, not liable to any such exception, but perfectly agreeable to that method of proof, which, from all learned antiquity, has been allowed of as perfectly clear and satisfactory. So that now, whether Sir *Isaac Newton*'s demonstration be admitted as conclusive or not, the truth of his rule at least is confirmed and established beyond all doubt or cavil.

The author of this *Discourse*, upon perusal of my *second Letter*, perceiving the sense, in which I had explained a passage or two of Sir *Isaac Newton*, to be a little different from the interpretation he had given, and perhaps meeting with persons who were of opinion, that I had hit upon the genuine meaning of these passages, has taken occasion, in giving a very modest and just account of his own work, in the *Republick of Letters*, for *October* last, to intersperse some observations upon the controverted passages, in order to assert his own interpretation, and to overthrow that which I had given.

To this piece, (for it is now become necessary to speak out, the dispute between us being hardly to be carried on any farther without quoting one another's words,) I endeavoured to give a clear and full *Answer* in every particular, in the *Republick of Letters* for the following Month. And in some points in question, it should seem, my answer gave satisfaction, or at least those points were not thought to deserve any longer dispute.

But to the rest of my *Answer* a *Reply* was published in the *Republick of Letters* for *December*: And this *Reply* I now purpose to examine, so far as I apprehend myself to be concerned in it.

It is insinuated, that between * considering evanescent augments as being actually vanished, and as being of any real magnitude, I have supposed that there can be represented to the mind some intermediate state of these augments at the very instant in which they vanish.

Now, as I cannot recollect, nor upon the most careful retrospection into the three pieces I have published upon this Subject, any where discover, that I have made, or given cause

* Republick of Letters, *Decemb.* 1735. *p.* 437, 438.

to think I made, the supposition here imputed to me, or any thing like it; nor so much as that I have ever joined this word STATE with evanescent augments, or with any such like expression; I could heartily wish, that learned Writer had pointed out the passage, where he imagines this supposition to be contained; or at least had told us the meaning of this word STATE, which now I can only guess at, the word being his, and none of mine.

If by this word is meant the same thing as magnitude, or quantity; as I am greatly inclined to think, from considering what is said in page 446 of this *Reply*, that the *conception of a quantity less than any whatever, has been thought possible*, which words seem manifestly to relate to me; then the supposition I am charged with, will stand thus: Between considering evanescent augments as being actually vanished, and as being of any real magnitude, there can be represented to the mind some intermediate magnitude of these augments at the very instant in which they vanish.

Now, to shew, that this could be no supposition, or opinion of mine, it will be sufficient to quote the following passages.

* "In the first place, and above all, it is here to be diligently attended to, that Sir *Isaac Newton* no where settles or determines the magnitude of nascent or evanescent increments, any farther than to say it is less than any finite quantity. On the contrary, he expressly declares, that their magnitude *cannot be assigned or determined.* Nor indeed has he any occasion to determine their magnitude but only the proportion between them, this being all that is requisite in his method.

"Now the proportion between two evanescent increments is easily to be conceived, tho' the absolute magnitude of these increments is *utterly imperceptible to the imagination.* For, *&c.*

† "I have no business at all to consider the magnitude of a moment. ‡ *Neque enim spectatur*, says Sir *Isaac Newton, magnitudo momentorum, set prima nascentium proportio.* I may tell you farther, that the magnitude of a moment is nothing fixed or determinate, is a quantity perpetually fleeting and altering till it vanishes into nothing; in short, that it is *utterly unassignable.* § *Dantur ultimæ quantitatum evanescentium rationes, non dantur ultime magnitudines.*

¶ "Vanishing quantities, though we can easily pursue them, and, as it were, keep them in sight, all the time that they are considered as having a finite magnitude and gradually decreasing, yet, when they arrive at the point of evanescence, do at once *slip away and withdraw themselves from our conception.*"

I might here produce several other passages, were not these already cited, abundantly sufficient to shew, that an intermediate magnitude of evanescent augments between their being actually vanished, and being of a real magnitude, cannot, in my opinion, be *represented to the mind.*

But, if by the word STATE be understood the proportion of the evanescent augments, which proportion I supposed to be different in the point or instant of evanescence, from what

* Minute Mathematician, *p.* 24, 25.

† Ibid. *p.* 55, 56.

‡ Princip. Lib. II. Lemm. 2.

§ Princip. Lib. I. Sect. 1. Schol.

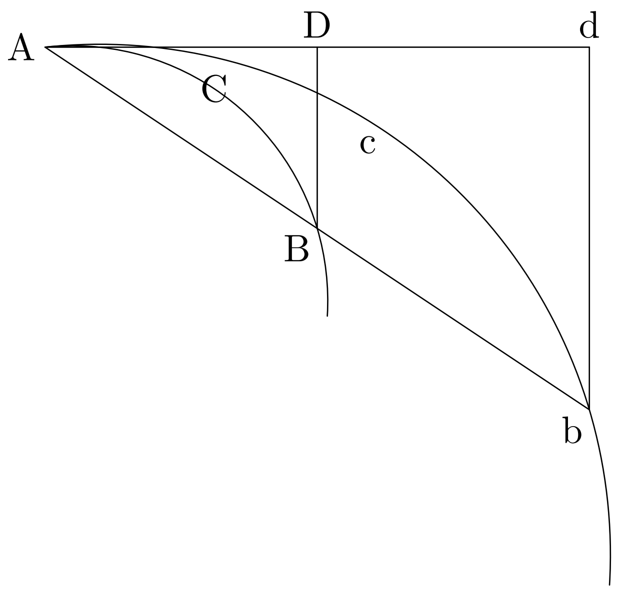
¶ Republick of Letters, *Novemb.* 1735. *p.* 378, 379.

it is before that instant, when it is perpetually varying, or after that instant, when it is none at all; I freely own, that such an intermediate *state* may, in my opinion, be *represented to the mind.* And I apprehend, I have clearly and distinctly shown, how it may be *represented to the mind*, and conceived, *viz.* * by contemplating this proportion, not in the vanishing quantities themselves, but in other quantities permanent and stable, which are always proportional to them.

Though by considering the entire passage, from which these last words are taken, as likewise *pag.* 383, 384 of the same discourse, and also what hath been said in my *second Letter, pag.* 25, 85, 86, I judge it to be very easy, for an unprejudiced and impartial reader, perfectly to understand what I mean by the proportion of nascent or evanescent augments: yet to prevent all mistakes as much as possible, I shall here once for all explain myself in such a manner, as I am persuaded, this learned Gentleman will not except against.

Let A and E be two flowing quantities, and let $\dot{a}$ and $\dot{e}$ be the respective contemporaneous augments of those flowing quantities, which augments let us conceive to be at first finite, but gradually decreasing, and at last vanishing into nothing at the very instant or point of time, that the flowing quantities arrive at the exact magnitudes A and E.

Likewise, let a and e be two finite quantities; one of which may be determinate and fixed, and the other may vary in such manner, that the quantities a and e shall always be proportional to the decreasing quantities $\dot{a}$ and $\dot{e}$, so long as $\dot{a}$ and $\dot{e}$ exist, and shall arrive at some determinate proportion, at the instant of time that those quantities $\dot{a}$ and $\dot{e}$ vanish into nothing, and cease to exist; in like manner as the finite lines Ad, Ab, in the seventh *Lemma*, arrive at the ratio of equality, at the instant of time that the decreasing quantities A D, A B, vanish into nothing, by the coincidence of the points B and A.



Then I say, at the point of time that $\dot{a}$ and $\dot{e}$ vanish, the velocities with which A and E flow, are proportional to the finite quantities a and e, which continue and subsist under that determinate proportion.

* Republick of Letters, *Novemb.* 1735. *p.* 379.

Thus much I think, must and will be granted to me: And this determinate proportion of the finite quantities a and e, is what I understand by the proportion of the evanescent augments. In which I apprehend, I exactly follow Sir *Isaac Newton*, as will be plain to any one who compares his words, * *Fluxiones, ut accurate loquar, sunt in prima ratione augmentorum nascentium*, or, † *Eodem recidit, si sumantur fluxiones in ultima ratione partium evanescentium*, with *Lemm.* 7, 8, 9, *Libr.* I. *Princip.*

But the point, in which I find the least likelihood of agreeing with this Gentleman, is the meaning of the first *Lemma Libr.* I. *Princip.* and consequently of the second and succeeding *Lemmata*, which depend upon the first, and must be explained by it.

This first *Lemma* therefore, as it is of great consequence to the understanding of Sir *Isaac Newton*'s doctrine, is what I have taken pains particularly to explain in my ‡ *second Letter*, and more fully in the § *Republick of Letters* for *November*. In both which places, I have distinctly proposed the several suppositions made in this *Lemma*, and the conclusion, which Sir *Isaac* draws from those suppositions. That conclusion is, That the quantities do *at last become equal*.

I could wish, the author of the *Discourse upon Fluxions* had been pleased to anatomise this *Lemma*, as carefully as I have done. This would have given me an opportunity of satisfying my self more clearly and perfectly, than I can now do, whether the conclusion of the *Lemma*, as he translates it, that the quantities *are ultimately equal*, be, in the sense he understands that conclusion in, at all different from one of the suppostions, *viz. That the quantities approach nearer than any given difference.*

If we say this is Sir *Isaac*'s conclusion, we must charge him with first supposing what he would prove, and with proving only what he has before supposed. But this, surely, is what that *Great Man* could not be guilty of.

Besides, his express words, which are as plain, as distinct, as little liable to ambiguity, as words can possibly be, *fiunt æquales*, do absolutely subvert such interpretation. These words, when used by a Mathematician, can imply nothing but equality, mathematical, *i. e.* absolute, perfect equality. Thus I have always understood, and always expressed them; and I am at a loss to find any good reason, why this Gentleman, in his *Reply*, should perpetually change the word *equal* or *equality*, which I constantly use in speaking of the first *Lemma*, for *coincidence* and *coinciding*, which I have not once made use of on occasion of that *Lemma*, nor indeed have any where mentioned above once or twice, and that only in treating of the second and seventh *Lemma*'s, where the word *coincidence* properly comes in.

I need not notice, that *coincidence* is something more than *equality.* Every body knows it. But I must observe, that I am wrongfully charged with supposing, ¶ *that the accuracy of the demonstrations founded on this* Gentleman's own *doctrine does in reality depend on this coincidence*, and that *in his demonstrations that circumstance is ever necessary.* This is a supposition I have never thought of making. For, I freely own, that, in the manner this ingenious Writer defines and treats of prime and ultimate ratio's, neither *coincidence*, nor

* Introduct. ad Quad. Curv.

† Ibidem.

‡ Pag. 88, 89.

§ Pag. 370, *&c.*

¶ Repub. of Lett. Dec. 1735. p. 440, 441.

equality, are at all necessary. His demonstrations are just and accurate without them. But his method is, I think, somewhat different from the method of Sir *Isaac Newton*. And if we will follow Sir *Isaac Newton*, it seems necessary to suppose a perfect equality, and sometimes a perfect coincidence, as in the second and seventh *Lemma*.

In order to make this necessity more manifestly appear, it may be proper once again to examine this first *Lemma* a little more critically. For which purpose I shall here repeat the *Lemma* it self, and shall then subjoin the translation of it, which this Gentleman calls a literal one and lastly a translation of my own.

LEMMA.

Quantitates, ut & quantitatum rationes, quæ ad aequalitatem tempore quovis finito constanter tendunt, & ante finem tempore illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo æquales.

Thus translated:

Quantities, and the ratio's of quantities, that during any finite time constantly approach each other, and before the end of that time approach nearer than any given difference, are ultimately equal.

I translate the *Lemma* thus.

Quantities, as also ratio's of quantities, that during any finite time constantly tend to equality, and before the end of that time approach nearer to one another than to have any given difference, do at last become equal.

I forbear making any remarks upon the difference between these two translations. But I must of necessity beg leave to ask one question. What is the meaning of these words, *are ultimately equal*? To this I can no where find a satisfactory answer.

It is said indeed, in the *Discourse upon Fluxions and ultimate Ratio's*, that * *the fixed quantity is* CONSIDERED *as the quantity, to which the varying quantity will at last or ultimately become equal*. And the † *limit of any ratio is here* CONSIDERED *as that with which the varying ratio will ultimately coincide*.

Also, in the account, or rather vindication, of that *Discourse*, in the *Republick of Letters* for *October* 1735, we are told, ‡ *the genuine meaning of the Lemma is, That the quantities are to be* ESTEEMED *ultimately equal, and the ratio's ultimately the same*. And yet we are assured, § *Sir* Isaac Newton *did not mean, that any point of time was assignable, wherein these varying magnitudes would become actually equal, or the ratio's really the same; but only that no difference whatever could be named, which they should not pass.*

And in the *Republick of Letters* for *December* last it is affirmed, that ¶ *variable quantities, or ratio's, and their limits, tho' they do never actually coincide, may come within the description of the Lemma; since the difference being once assign'd*, or, as is said a little before,

* Page 48.
† Ibid.
‡ Page 254.
§ P. 255.
¶ P. 442.

first assign'd, the approach of these quantities may be so accelerated, that in less than any given time the variable quantity, and its limit, shall differ by less than the assign'd difference.

But still I am at a loss to conceive, how quantities, which do never *become actually equal*, can by a Mathematician be *considered as equal*, or *esteemed equal*, or can *come within the description of a Lemma*, which *Lemma* expresly affirms, that they *become equal*.

In this difficulty, I met with no other assistance, than what arises from part of the two passages last cited, *That no difference whatever can be named, which the quantities shall not pass*; and, That *the difference being first assign'd, the approach of these quantities may be so accelerated, that in less than any given time the variable quantity, and its limit, shall differ by less than the assign'd difference.*

Things standing thus, it may be proper to review the *Lemma*, when cleared up by this, *the true interpretation.* I think it will run to this effect.

Quantities, and the ratio's of quantities, that during any finite time constantly approach each other, and before the end of that time approach nearer than any difference first assigned, shall differ by less than the assigned difference.

And here a new difficulty arises, as great as the former, for I cannot possibly find, that the conclusion contains any thing more than the supposition, or condition, laid down in order to come at that conclusion.

Let us try therefore, if, by a careful enquiry, we cannot find a more genuine interpretation of these words, *fiunt ultimo æquales.*

The first of these is *fiunt, become*; not *sunt, are.* It signifies therefore that the quantities, or ratio's, *become* what they were not before.

The next is *æquales, equal.* And *equal*, with any good writer, but especially in the mouth of a Mathematician, signifies perfectly, exactly equal, being without any difference. The quantities, or ratio's, do therefore become perfectly equal. It only remains to see *when* they become so.

This is expressed by the word *ultimo, at last.* And we are next to examine, what time, or what point of time is hereby denoted. Which will be easily found by considering the preceding words of the *Lemma.*

For the quantities, or ratio's, are supposed to tend constantly to equality, during a finite time.

Therefore their tendency to equality does not cease, during that time, they do not become perfectly equal, during that time, i. e. *before* the end of that time.

Nor is their tendency to equality supposed to continue beyond that time. Consequently, they cannot become equal *after* the end of that time.

It follows therefore, that they become equal *at* the end of that time, at the instant of the expiration of that time, and at no other.

Having now fixed the meaning of the word *ultimo* in the *Lemma*, it will easily be allowed me, that this word must have the same meaning in the demonstration, in the words, *fiant ultimo inæquales*, let them become unequal at the expiration of the finite time. Nor can it be disputed, but that the word *ultima*, in the expression *ultima differentia*, denotes the same point of time; and consequently that *ultima differentia, the last difference*, signifies the difference at the expiration of the given time, when the tendency to equality is entirely over.

It may be proper also to observe, that the having a *last difference* is, in the demonstration, asserted to be contrary to the supposition, of the quantities approaching nearer together than any *given difference.* This *last difference* therefore is by Sir *Isaac Newton* understood to be equivalent to the *given difference* in the *Lemma.* Consequently, that *given difference* is not a *difference first assigned, according to which the degree of approach of those quantities may be afterwards regulated*, but is a *difference that may be taken, after the celerity or degree of approach of those quantities is in every part determined.* And in this case it is * granted me, that a perfect equality may be intended in this *Lemma.*

The indisputable explication of *ultima differentia* above given, shews it to be utterly impossible to apply this *Lemma*, or the demonstration of it, to the case of the ordinate to the *Hyperbola*, and the same ordinate continued to the *Asymptote*; or to two quantities having at first a given difference, and increasing by equal additions *ad infinitum.* If any one thinks otherwise, I should be glad to see such application particularly and distinctly made.

The first *Lemma* being thus fully cleared up, we shall thereby be better enabled to enter upon the consideration of the second, where we have two points to examine.

I. In what sense Sir *Isaac Newton* designs this *Lemma* should be understood.

II. The truth of the *Lemma.*

I. That Sir *Isaac* intends a perfect equality of the inscribed, circumscribed, and curvilineal figures, appears to me inevitably demonstrable from the following considerations.

1. The second *Lemma* is deduced from the first, in which, it has been already shewn, that a perfect equality is intended.

2. At the close of the demonstration of the second *Lemma*, he uses these words, *fiunt ultimo æquales*, which, as we have before observed, will admit of no dispute.

3. In the third *Lemma*, which differs no otherwise from the second, than that the breadth of the parallelograms are now made unequal, he tells us, in the first corollary, that these figures do *coincidere omni ex parte*, words as strong as can be used, to express an absolute equality.

4. In the fourth corollary to this third *Lemma*, he affirms these last figures, or the inscribed and circumscribed figures in their last form, not to be rectilineal, but to be curvilinear limits of the rectilineal figures; *i. e.* The inscribed and circumscribed figures, in their last form, coincide with, or degenerate into, the curvilineal figure.

From all these particulars it appears to me to be as clear as the day, that, in this second *Lemma*, Sir *Isaac Newton* intended a perfect equality and exact coincidence, of the inscribed and circumscribed, with the curvilineal figure.

But it is said that † "the quantities in many of the succeeding *Lemma*'s, to which the first is applied, are such where the approach is determined by a subdivision into parts; but by this method of proceeding it is obvious, that no coincidence can ever be obtained."

And a little after, ‡ "supposing the coincidence could by this means (by motion) take place, it would prove that no such coincidence was ever intended by Sir *Isaac Newton*; since had he regarded it as a necessary circumstance, he would certainly have applied to this *Lemma* a method of inscribing the figure, by which such a coincidence might be shown; whereas by describing the parallelogram by a continual division, and making their bases constantly equal,

* Rep. of Lett. Dec. 1735. p. 442.

† Republick of Letters, *Decemb.* 1735. *p.* 442.

‡ Ibid. *p.* 443, 444.

and always some *aliquot* part of the whole, he has necessarily excluded the description of them by motion, by which means only it is supposed that this coincidence can be brought about."

In answer to this, I desire it may be observed, that Sir *Isaac* gives us two different cases of inscribing the parallelograms. In the first case their bases are all equal, and this case is the subject of the second *Lemma*. In the second case, the bases of the parallelograms are unequal, and this second case is the subject of the third *Lemma*.

Now, by considering these two *Lemmata* together, I think it may appear,

1. That Sir *Isaac Newton* has applied to these *Lemmata* a method of inscribing the figure, by which such a coincidence may be shown.

2. That he does not describe the parallelograms by a continual division, or subdivision into parts.

3. That he does not make their bases constantly equal, and always some *aliquot* part of the whole.

For, 1. The method taken by Sir *Isaac* to inscribe the figure, is, in the first place, to erect any number of parallelograms taken at pleasure, either on equal bases, as in the second *Lemma*, or unequal, as in *Lemma* the third. The next step he takes, is to suppose the bases of the parallelograms to be diminished, and their number to be encreased *ad infinitum*. From which he proves, by virtue of the first *Lemma*, that the figures under consideration do at last become equal.

2. Sir *Isaac Newton* says not one word of continual division, or subdivision into parts, of the base of the figure, in order to describe the parallelograms. And his words of diminishing the bases and encreasing their number *ad infinitum*, certainly are not more applicable to a continual division, than to lessening the bases by a continued motion. But this latter method is agreeable to his constant practice, and fully answers his purpose, which the former will not. Equity therefore requires, that his words should be taken in the latter sense.

3. As in the third *Lemma* the bases are made unequal, it is plain Sir *Isaac Newton* does not make these bases CONSTANTLY equal, nor ALWAYS some *aliquot* part of the whole.

It does not appear therefore, that Sir *Isaac* has excluded the description of the parallelograms by motion.

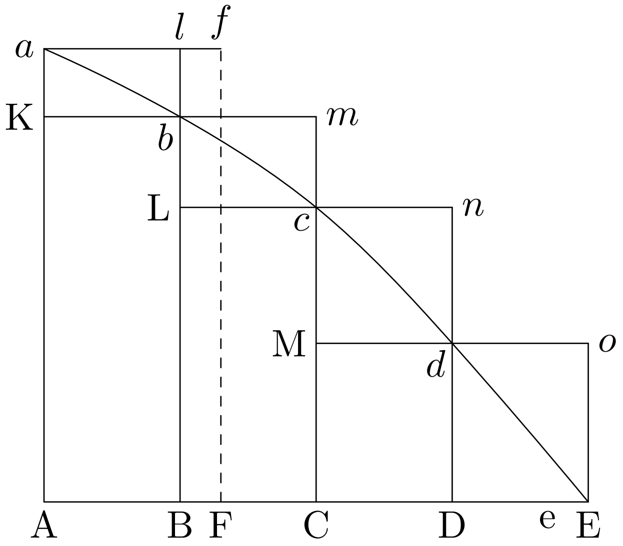
Having now removed all the objections that have been made against my assertion, that Sir *Isaac Newton* in this *Lemma* intended a perfect equality and coincidence, of the inscribed and circumscribed figures, it is time to come to the second head of our proposed enquiry.

II. The truth of the *Lemma*, when understood in the sense of a perfect equality and coincidence of the figures.

This may be seen distinctly and particularly made out in the *Republick of Letters* for *November*, *p.* 376, 377; where it is shown, how, by motion, all the suppositions of the first *Lemma* may be introduced into this second *Lemma*; and consequently that the perfect equality demonstrated in the first, must take place in this *Lemma* also.

This equality therefore we are obliged to acknowledge, although we should not be able, by stretch of imagination, to pursue these figures, and, as it were to keep them in sight all the way, till the very point of time that they arrive at this equality. For a demonstrated truth must be owned, though we do not perfectly see every step by which the thing is brought about.

It is sufficient to be assured, that the rectangle A B l a is the difference between the inscribed and circumscribed figures, so long as these figures are rectilineal; and that within a given time this rectangle; or difference, will grow less than any assignable quantity. When we are certain of thus much, we may depend upon the first *Lemma* for the rest, that being a truth before demonstrated.



It will then be only a kind of illustration, but no way necessary to proving the truth and certainty of the *Lemma* in question, if we say that, as by the motion of the point B towards A, A B the base of the rectangle must perpetually diminish; so by the arrival of B at the point A, that base, and with it the rectangle, *i. e.* the difference between the inscribed and circumscribed figures, will entirely vanish; by which means these two figures will perfectly coincide with one another, and with the curvilineal figure.

We have therefore no occasion for *the delineation of a line less than any line that can be assigned.* We acknowledge such delineation to be utterly impossible; as likewise the *conception* of such a line, as an actually existing, fixed, invariable, determinate quantity. But then, on the other hand, it cannot be denied, that a line constantly diminishing by motion, and at last vanishing, must necessarily grow less than any line that can be assigned, though we cannot fix a point of time when it is so, till it is actually vanished and become nothing.

But were it possible to conceive, or even to delineate such a line, I do not see it could be of any service to us upon this occasion.

It would not enable us *to describe a figure within the curve under the suppos'd circumstance of coincidence.* For a figure coinciding with the curvilinear figure cannot be within the curve.

Nor would it help us in *describing the last form* of the inscribed or circumscribed figure; since that last form is not rectilineal, but curvilineal.* *Hæ figuræ ultimæ* (*quoad perimetros* acE) *non sunt rectilineæ, sed rectilinearum limites curvilinei.*

The perimeter of the circumscribed figure, so long as it continues rectilineal, is indeed equal to the sum of the lines a E, A E: but when that figure, at the instant that the rectangle

* Princip. Lib. I. Sect. 1. Lemm. 3. Coroll. 4.

A B l a vanishes into nothing, does thereby come to coincide with the curvilineal figure, as its last form, or its limit which it then arrives at; its perimeter is then no other than the curve itself.

Some mention having been made of *aliquot* parts, it may now be necessary to insert a remark, which when I drew up my last paper, I thought too obvious, and indeed too trifling, considering Sir *Isaac Newton*'s third *Lemma*, to trouble the reader with. Suppose the line A B, at any point of time, to be an *aliquot* part, for instance one fourth, of the line A E: and let the point B move a very little further towards A. Then will A B, and the lines equal to it, as B C, C D, D e, be each them a little less than one fourth of A E; and consequently, besides these four lines, there will be left, adjoining to the point E, a fifth line *e*E smaller than any of the rest. But since, in this case, the base A B will still be the greatest breadth, as in the third *Lemma*, it is plain that the rectangle A B l a will be greater than the difference between the figures; and consequently, when this rectangle vanishes, the whole difference between the figures must vanish with it.

I am very free to own, *that Sir* Isaac Newton *does not* ALWAYS *consider this coincidence*, or rather equality, *of the variable quantity, or ratio and its ultimate, as necessary in his method.* I have already taken notice, † that, if two lines increasing without limit have always a given difference, their ratio will never be absolutely equal to the ratio of equality, the *ultima ratio*, or limit, to which their variable ratio perpetually tends, but which it never arrives at. I likewise readily agree, that, where the expression *ultima ratio* is used, it always signifies a limit, to which a variable ratio constantly tends, and to which it approaches nearer than any given difference. But it does not always signify, as in the case just now mentioned, a limit which the variable ratio never arrives at.

It has been affirmed indeed, that it *cannot possibly be suppos'd in this place to have a signification different from what it had in the first and subsequent Lemma's.* And if this be true, it will follow, that, according to Sir *Isaac Newton, the ultimate ratio is a ratio that the variable one it is ascrib'd to can* NEVER *coincide with.*

It is therefore necessarily incumbent upon me to examine, whether the *phrase* ultima ratio, *peculiar to Sir* Isaac Newton *and his Method*, has not *in this place a signification different from what it had in the first and subsequent* Lemma's.

In the first place, it is agreed on both Sides, that in this case the ultimate ratio is one that the variable ratio cannot coincide with. Let us see if it be thus in the *Lemma*'s.

In *Lemma* 1. I do not meet with the phrase *ultima ratio*: I only find the expression *fiunt ultimo æquales*, the sense of which we have already considered. In the second and third *Lemmata* we meet with the words *rationes ultimæ*, and we find them expounded by the clear and decisive expressions already mentioned; *fiunt ultimo æquales*; *coincidit omni ex parte*; *coincidit ultimo*; and, *figuræ ultimæ non sunt rectilineæ, sed rectilinearum limites curvilinei*; which evidently shew, that *ratio ultima* has, in these *Lemma*'s a signification somewhat different from what it bears in the case above cited. It always signifies a limit: But, in them, denotes a limit which the variable ratio punctually arrives at; and, in the other, a limit which the variable ratio can never arrive at.

† Rep. of Lett. for Nov. 1735. *p.* 375, 381, 382.