CONSIDERATIONS UPON SOME PASSAGES CONTAINED IN TWO LETTERS TO THE AUTHOR OF THE ANALYST

By

James Jurin

(The Present State of the Republick of Letters, November 1735, pp. 369–396)

Edited by David R. Wilkins

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NOTE ON THE TEXT

This text is transcribed from *The present state of the republick of letters* for November 1735.

In the first figure, the vertex labelled 'f' is labelled 'c' in the original article. This would seem to be a typographical error: another of the vertices of the diagram is also labelled 'c', and moreover this figure is intended to reproduce that accompanying *Lemma II*, *Lib. I* of Newton's *Principia*, where this vertex is labelled 'f'.

The following spellings, differing from modern British English, are employed in the original 1735 text: throughly, rendred.

David R. Wilkins
Dublin, June 2002

CONSIDERATIONS *upon some passages contained in two Letters to the Author of the* Analyst*, written in defence of Sir* Isaac Newton*, and the* British *Mathematicians. By* Philalethes Cantabrigiensis.

[*The Present State of the Republick of Letters*, November 1735, pp. 369–396.]

IN defending Sir *Isaac Newton*, I thought proper to keep as close as possibly I could, not only to the turn of thought, but even to the phrase and manner of expression of that incomparable Writer. He appeared to me to have so throughly considered those parts of his works, especially, in which the foundation of his doctrine is laid down, to have weighed every word of them with so much accuracy and caution, and to have delivered himself with such wonderful perspicuity, to a reader who peruses him with the necessary attention, that if I did not despair of seeing the things he treated of, better expressed by any Writer whatsoever, at least I thought it utterly impossible for me to do so. Besides, what I had undertaken, was to defend what Sir *Isaac Newton* had written, not to alter it, much less to mend it.

This caution of mine, I reasonably hoped, would not have been disagreeable to any of those, whom I was endeavouring to defend against the attacks of a conceited and petulant Writer. But so it is: I find some Persons, and such too as have greatly distinguished themselves in this part of Learning, are of opinion, that instead of following my Author so closely, I should have done better, and have come nearer to his doctrine, and to Truth, if I had deviated a little from his expression, which they esteem to be sometimes not so accurate as were to be wished. In this they must pardon me. I pay a great deference to their judgment; but a greater to that of Sir *Isaac Newton*; and a greater still to Truth. These Gentlemen seem likewise to think, that I have not always hit upon the true sense of my Author. And in this I shall pay them that respect, to make their objections the subject of a strict and careful examination. They may, I think, be all reduced to the three following heads.

I. My explication of *Lemma* 1. *Lib.* I. *Princip.*

I. The sense of the *Scholium ad Sect* 1. *Libr.* I. *Princip.* particularly as to, 1. The doctrine of Limits, 2. The meaning of the term evanescent, or vanishing.

The sense of *Lemma* 2. *Lib.* II. *Princip.*

I. As to the first of these, if I comprehend the Objection rightly, it is briefly this, That by the words *fiunt ultimo æquales*, I understand, that the quantities do at last become equal. Now, whether Sir *Isaac Newton*'s words are by me truly rendred into *English*, may be left to the consideration of every Reader. And whether what Sir *Isaac* here says, be exactly agreeable to truth, will be judged of by those, who have throughly weighed his demonstration.

To me it appears, that Sir *Isaac* means by the words of the *Lemma*, and proves in his demonstration, not that the quantities, or ratio's, are barely to be considered as ultimately becoming equal, or are to be esteemed as ultimately equal, though, in reality, they can never have that proportion to each other; but that they do at last become actually, perfectly, and absolutely equal. And this, I think, will be evident from the *Lemma* itself, from the demonstration of it, and from the use that is made of it in demonstrating the subsequent *Lemmata*, particularly the second, third and fourth. The *Lemma* itself runs thus.

Quantitates ut & quantitatum rationes, quæ ad æqualitatem tempore quovis finito constanter tendunt, & ante finem temporis illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo æquales.

I have already observed (*Letter* 2., *p.* 88, 89.) that in this *Lemma* are contained the four following suppositions.

1. That the quantities, or ratio's, constantly tend to equality, *ad æqualitatem constanter tendunt*,

2. During some finite time, that either happens to be determined in any particular case, or else may be proposed and assumed at pleasure, *tempore quovis finito*,

3. And come nearer together than to have any given difference, *& propius ad invicem accedunt quam pro data quavis differentia*,

4. Before the end of that finite time, *ante finem temporis illius.*

If any one of these suppositions be omitted; much more if the second and fourth be left out; or if, with the Author of the *Analyst*, we neglect the first, second and fourth; we can never justly come at Sir *Isaac*'s conclusion, That, at last, *i. e.* at the end of the given time, during which the quantities, or ratio's, were supposed to tend constantly to equality, they become equal.

To make this perfectly evident, let us illustrate it by a familiar and easy example of a variable quantity becoming ultimately equal to a fixed quantity, upon the foot of these four suppositions. First, for a fixed quantity, let us take so much of the circular arch of a clock, described by the extremity of the hand, as is intercepted between the hour lines of 11 and 12. And let our variable quantity be such part of the same circular arch, as the extremity of the hand begins to describe at 11 o'clock, and continues to describe during an exact hour of time. Now, let us suppose the clock to go so well, as that, during an exact hour of time, the hand shall not pass the hour-line of 12; but shall approach that hour-line within a distance less than any assignable, before the end of the hour. Then will these two circular arches,

1. Constantly tend to equality,

2. During an hour,

3. And will come nearer to one another than to have any given difference,

4. Before the end of the hour.

From which it follows, that at the end of the hour, the two quantities must become equal. For if they are not equal at the end of the hour, they must have some difference. And the point of time being given, the difference must then actually exist; and must exist not variable and changing, as while the two quantities were tending to equality; but permanent and fixed, the tendency to equality being now over; and consequently the difference must be determinate and given. Therefore the two quantities did not come nearer together than that given difference before the given point of time, or before the end of the hour. Which is contrary to the fourth supposition.

Here I crave leave to make a few Observations.

1. By taking in the consideration of a finite time, Sir *Isaac Newton* is able to assign a point of time, at which he can demonstrate the quantities to be actually equal.

2. Thereby he greatly assists the imagination, in conceiving how the equality is brought to pass. For as he supposes a difference perpetually to subsist, before the expiration of the given time; the mind of the attentive reader, in contemplating that difference, contemplates likewise the remainder of the given time corresponding to that difference; and easily conceives, that as the variable remainder of the time is gradually wearing away, the variable difference, by the first supposition, must wear away with it; and consequently, that they may both vanish together. On both these accounts, this method seems greatly to exceed the method of the ancients, in perspicuity, as well as in the conciseness of the demonstrations, though this last alone is to me an inestimable advantage. For *Ars longa, vita brevis.*

3. It is not the meaning of Sir *Isaac*'s conclusion, *fiunt ultimo æquales*, that no difference whatever can be named, which the quantities shall not pass. For that is not his conclusion; but is his third supposition, which, together with the rest, is laid down in order to draw his conclusion from it, That, at last, *i. e.* at the expiration of the given time, when the quantities no longer tend to equality, they become equal. In like manner, having shown, in his demonstration of *Lemma* the eleventh, *Cas.* 1. that two ratio's may come nearer together than to have any given difference, he thence infers, by virtue of this first *Lemma*, that these ratio's do at last become equal. Likewise, in the same *Lemma, Cas.* 3. having taught us, that two angles may approach nearer to equality, than to have any given difference, he thence concludes, by the same first *Lemma*, that those angles at last become equal.

4. To shew the strict sense, in which Sir *Isaac Newton* understands his quantities to become equal, it may be proper to observe, that although the ordinate to a diameter of the Hyperbola be understood, by some other geometers, to become at last equal to the same line continued to the Asymptote; yet the equality can never be proved by this *Lemma*. On the contrary, the *Lemma* is, with great judgment, so worded, on purpose, as necessarily to exclude this and such like cases.

For, though these two lines, *viz.* the ordinate ending at the curve, and the same ordinate continued to the Asymptote, do,

1. Constantly tend to equality,
2. During any finite time that can be proposed,
3. And come nearer to each other than to have any given difference: yet,
4. It cannot truly be said, that they come thus near before the end of a finite time.

For, let the time proposed be what it will, and let the velocity, with which the diameter flows, during that time, be ever so great; yet that diameter will be given at the end of that time; and consequently, both the ordinate, and the part intercepted between the curve and the Asymptote, *i. e.* the difference of the two lines, will be given at the end of that time; much more at any instant before the end of that time. So that the fourth supposition in the *Lemma* will by no means suit this case, and consequently the *Lemma* is not applicable to it. Nor can the demonstration of this *Lemma* possibly be applied to this case; since these two variable quantities can have no last magnitude, and consequently cannot be supposed to have any last difference.

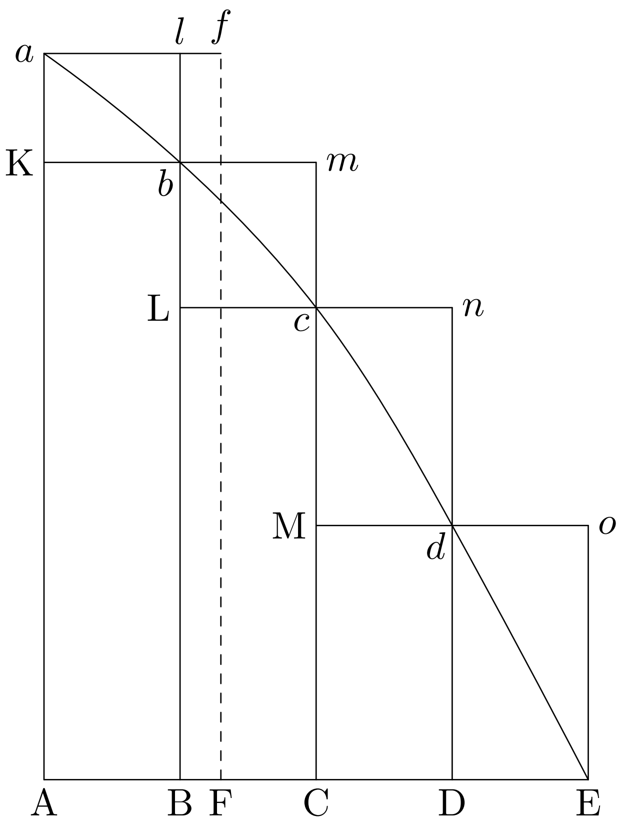
Much less can this *Lemma* be applied to two lines increasing together by equal additions, or with equal velocities, *ad infinitum*, and having from the first a given difference. For that difference always continuing the same, it is plain these lines cannot come nearer together than to have any given difference, as they ought to do, by the third supposition, in order to come within the sense of this *Lemma*.

If it be said that, although the lines themselves cannot approach to one another within less than a given difference, yet the ratio of the two lines will approach nearer to the ratio of equality than any given difference: I agree it will do so; but not in a finite time, as it ought to do by the fourth supposition.

But as Sir *Isaac Newton* has mentioned this case, I will endeavour to shew on what account, and in what sense he does so, when I come to the *Scholium*, wherein he speaks of it.

A second example, to illustrate and confirm the explication of this *Lemma*, may be found in that immediately following.

In the demonstration of which, I suppose my reader to have followed Sir *Isaac Newton* so far, as to see, that the rectangle A B l a is the difference between the inscribed and circumscribed figures; and likewise to conceive that, as the number of the rectangles is increased, and their breadth diminished *ad infinitum*, that rectangle A B l a must grow less than any given quantity. It remains therefore only to shew him, how this may be brought within the case of the first *Lemma*; which is the difficulty at which Mr. *Nieuwentiit* formerly, and others have since stumbled.



Let us imagine a point, as E, to describe the line E A, with a continued motion, in the space of an hour. And let it be conceived that, in every point of time during the hour, a rectangle, as A B l a, is raised upon A B, that part of the line E A, which at that point of time is yet undescribed. Also, upon every other part of the line, equal to A B, let other rectangles be erected, as in the figure, at the same point of time.

It is manifest, that the magnitude of each of these rectangles will be perpetually diminishing, and their number perpetually increasing; and consequently the inscribed and circumscribed figures will constantly approach nearer together, during the hour.

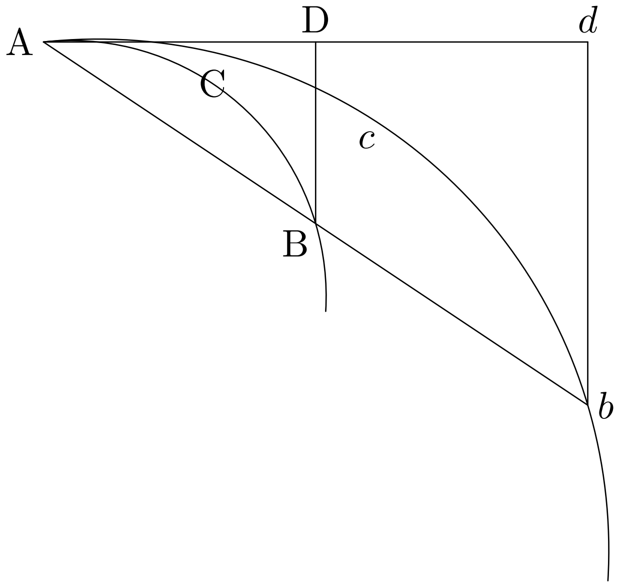
Also, since we have supposed the whole line A E to be described during the hour, it is evident that, before the end of the hour, the part undescribed B A must grow less than any given quantity. Consequently, the rectangle A B l a must grow less than any given quantity before the end of the hour, *i. e.* the difference between the inscribed and circumscribed figures must grow less than any given quantity before the end of the hour. Then we have introduced into the case before us all the suppositions of the first *Lemma*. For

1. The two figures tend constantly to equality,
2. During one hour,
3. And come nearer to one another than to have any given difference,
4. Before the end of the hour.

Therefore, by the first *Lemma*, they become equal at the end of the hour.

If any man shall say, that a right-lined figure, inscribed in a curvilineal one, can never be equal to that curvilineal figure; much less to another right-lined figure, circumscribed about the curve; I agree with him. I am ready to own that, during an hour, these figures are one inscribed, and the other circumscribed; that neither of them is equal to the curvilineal figure, much less one to the other. But then, on the other hand, it must be granted me, that, at the instant the hour expires, there is no longer any inscribed or circumscribed figure; but each of them coincides with the curvilineal figure, which is the limit, the *limes curvilineus*, at which they then arrive.

If what I have said upon this second *Lemma*, be expressed with sufficient clearness; it will not be necessary for me to enter into the consideration of the third and fourth. It will be easy for every reader to apply to them, so far as is necessary, what I have said upon this. But the seventh *Lemma* requires to be considered in a manner somewhat different.



In this *Lemma* Sir *Isaac Newton* has, with great skill and judgment, taken a method very different from what he had used in the other.

His design is to shew that, when the chord A B, the arch A C B, and the tangent A D limited by any secant B D, come to vanish by the coincidence of the points B and A, their last ratio is the ratio of equality. And in order to make this be more clearly conceived, he directs our imagination, not to these vanishing quantities themselves, but to others, which are proportional to them, and always preserve a finite magnitude. The former, tho' we can easily pursue them, and, as it were, keep them in sight, all the time that they are considered as having a finite magnitude and gradually decreasing, yet, when they arrive at the point of evanescence, do at once slip away and withdraw themselves from our conception. It is of great use therefore, to contemplate their proportion, not in themselves, but in other quantities permanent and stable, which are always proportional to them, as in the finite chord $\mathrm{A}\,b$, the finite arch $\mathrm{A}\,c\,b$, and the finite tangent $\mathrm{A}\,d$, limited by the finite secant $b\,d$, parallel to the former secant B D. Now, since he has proved, in the preceding *Lemma*, that at the instant of time, when the points A and B coincide, the angle B A D, or $b\,\mathrm{A}\,d$, will vanish; it is easy to conceive that, in this case, the chord $\mathrm{A}\,b$ must coincide with the tangent $\mathrm{A}\,d$, *i. e.* those two lines must become perfectly equal; and consequently the vanishing lines A B, A D must likewise, at the same instant of time, arrive at the same proportion of a perfect equality. Nor is it at all material to consider what angles the secant $b\,d$ makes with the lines $\mathrm{A}\,b$, $\mathrm{A}\,d$, before their coincidence, whether equal, or not. For when those lines coincide, the secant must make the same angle with them both.

This same artifice of representing to the imagination, finite and permanent quantities, instead of evanescent ones proportional to them, is continued in the 8th and 9th *Lemma*.

II. I come now to consider Sir *Isaac Newton*'s *Scholium* at the end of his *Methodus rationum primarum & ultimarum*, with regard to two particular points, in which I have the misfortune to differ from some other persons.

1. The doctrine of limits. 2. The meaning of the term evanescent, or vanishing.

1. In order to treat clearly of the first of these points, it will be proper, in the first place, to fix the signification of the word limit, and also what is meant by the expression, *attingere limitem*, to reach or arrive at a limit.

I apprehend therefore that, by the limit of a variable quantity, is meant some determinate quantity, to which the variable quantity is supposed continually to approach, and to come nearer to it than to have any given difference, but never to go beyond it. And by the limit of a variable ratio, is meant some determinate ratio, to which the variable ratio is supposed continually to approach, and to come nearer to it than to have any given difference, but never to go beyond it. By arriving at a limit I understand Sir *Isaac Newton* to mean, that the variable quantity, or ratio, becomes absolutely equal to the determinate quantity, or ratio, to which it is supposed to tend.

This being settled, it will be of use to take notice, that there are two cases of a variable quantity, or variable ratio, tending to such a limit, as we have been describing.

In the first case, the variable quantity, or ratio, will not only approach to its limit nearer than any given difference, *propius assequi limitem quam pro data quavis differentia*; but will actually arrive at its limit, will *attingere limitem*.

In the second case, the variable quantity, or ratio, will only approach its limit within less than any given difference; but will never actually arrive at it.

Now whether a quantity, or ratio, shall arrive at its limit, or shall not arrive at it, depends entirely upon the supposition we make of the time, during which the quantity, or ratio, is conceived constantly to tend or approach towards its limit.

For, 1. If we take a given time for this constant approach, and suppose that, before the end of that given time, the quantity, or ratio, shall approach its limit within less than any given difference; then, by the first *Lemma*, the quantity, or ratio, will actually arrive at its limit, will *attingere limitem*, at the expiration of that given time.

3. If we suppose this constant approach to continue longer than any assignable time, to continue without end, as in the boundless approximation of the Hyperbola to its Asymptote; then, although the quantity, or ratio, will approach its limit within less than any given difference, yet it will never arrive at it, it can never *attingere limitem*.

As in the method of first and ultimate ratio's, as well as in his *Philosophiae Principia*, and his doctrine of Fluxions, both which are founded upon this method, Sir *Isaac Newton* contents not himself with any the most near approximations, but carries his demonstrations to the utmost accuracy and Geometrick rigour; accordingly, every one of the examples he has given in the *Lemmata* of this section, are of such quantities, and ratio's, as actually arrive at their respective limits. Nor do I remember, that he has ever taken occasion, to give an instance of a quantity, or ratio, which never arrives at its limit, except one towards the latter end of the *Scholium* we are treating of, and even that by way of illustration only.

It is the instance I had before mentioned, of two quantities, having at first a given difference, and increasing together by equal additions *ad infinitum*, which is introduced upon this occasion. He had just before been guarding against an objection, which possibly might by inadvertent readers be sometime or other brought against his doctrine, as labouring under the absurdity of admitting indivisibles; a fault, which I am bold to say, he was never guilty of, and the bare suspicion of which he has endeavoured to prevent, both here, and in many other parts of his writings, with the utmost caution.

The objection was, that if the last ratio's of evanescent quantities could be assigned, the last magnitudes of those quantities might likewise be assigned.

He answers, No. For those last ratio's, with which the quantities vanish, strictly speaking, are not the ratio's of the last quantities, *i. e.* of the last determinate magnitudes of the quantities: But they are the limits, to which, while the quantities themselves decrease without limit, the ratio's of those quantities do perpetually approach; and to which those ratio's may come nearer than any given difference; but can never pass them; nor can arrive at them before the quantities are diminished *ad infinitum*.

The thing, says he, will be more clearly understood in quantities infinitely great. If two quantities, whose difference is given, be increased *ad infinitum*, their last ratio, *i. e.* their limit, will be given, to wit the ratio of equality: But the last, or greatest quantities, having that ratio will not be given.

Here it is easy to perceive, that Sir *Isaac* never imagined the ratio of these quantities to arrive at the limit, *attingere limitem*. If any one thinks otherwise, I would desire him to apply to this ratio the words which had been used immediately before, in speaking of the ratio's of evanescent quantities. Those, says Sir *Isaac Newton*, cannot arrive at their limit, before the quantities are diminished *ad infinitum*. By parity of reason, in this case, which is brought in only to illustrate the other, the ratio of the two increasing quantities cannot arrive at its limit, before the quantities are increased *ad infinitum*. But can they ever be really and in fact increased *ad infinitum*? Can they ever have a magnitude really infinite? Really greater than any quantity that can be supposed? No certainly. Neither therefore can their ratio arrive at its limit. But decreasing quantities may really and in fact be diminished *ad infinitum*: For they may vanish and come to nothing. The ratio therefore of these may arrive at its limit, though that of the other cannot. The same reasoning is easily applied to the case of the Hyperbola and its Asymptote.

2. The second doubt, that has arisen about the matter of the *Scholium*, relates to the sense, in which Sir *Isaac Newton* uses the word evanescent or vanishing.

That by the vanishing of any quantities, is meant their passing from existence to non-existence, from something to nothing, is not disputed. The only question is, whether the quantities that vanish, are understood to spend some finite time in vanishing, or to vanish in an instant, or point of time; and consequently, whether they bear to one another an infinite number of different successive ratio's, during the vanishing, or one ratio only, at the point, or instant, of their evanescence.

And here I must remark, that this dispute is of no other consequence, than only to determine, whether the sense, in which I use the word evanescent, or vanishing, confining it to an instant of time, be agreeable to the sense it is used in by Sir *Isaac Newton*. For, if the quantities vanish in an instant, and I take the only ratio with which they vanish; or they spend a finite time in vanishing, and I take the last of the ratio's, which they successively bear to one another during that time; still the ratio, taken in either of these cases, will be one and the same.

Farther, I am free to own that, if the quantities spend a finite time in vanishing, it is absurd to speak of the ratio, simply, with which they vanish. If I would be clearly understood, I must say the last ratio. But then, on the other hand, if their vanishing be confined to an instant of time, and consequently to one only ratio; it will be indifferent for me to say, the ratio with which they vanish, or the last ratio of the vanishing quantities. And my using indifferently sometimes the one, and sometimes the other, of these expressions, would, to a careful Reader, be a proof, that my intention was to confine the vanishing to a single instant. This proof therefore Sir *Isaac Newton* has furnished us with, that he uses the word in that sense: For though he often joins the word *ultima* to ratio, to give emphasis to his expression, and to excite the attention of his reader; yet at other times he speaks of the ratio, with which the quantities vanish, without the word *ultima*; as he does likewise of the ratio, with which the quantities begin to exist, without adding the word *prima*.

If this proof be not thought sufficient, I would desire my Reader attentively to read over the comparison in this *Scholium*, between the last velocity of a body arriving at a certain place, where its motion ceases, and the last ratio of evanescent quantities: And when he has carefully considered these words in particular, *neque antequam attingit locum, neque postea, sed tunc cum attingit*; as also the words *illam ipsam velocitatem quacum corpus attingit locum ultimum*, in the former case; and in the latter, *intelligendam esse rationem quantitatum, non antequam evanescant, non postea, sed quacum evanescunt*; if he is not then of opinion, that the words *attingit* and *evanescunt*, do both imply one single instant, or point of time, I will not undertake to give him a plainer proof.

But if any one shall say, that although the verb vanish implies only an instant of time, yet the participle vanishing must be undertood to relate to some finite time, I can only reply, that such an assertion seems to me to contradict the common use of language. And however this matter be taken, as I have carefully followed Sir *Isaac Newton*'s expression, it is but reasonable to hope, that the same words, when used by me, will be equally allowed of, as when they are made use of by him.

III. The sense of *Lemma* 2. *Libr.* II. *Princip.*

As the design of this *Lemma* is to determine the proportion between either the moments, or the velocities of flowing quantities, upon which proportion the whole doctrine of fluxions depends, it will be necessary in the first place to settle the signification of the word moment.

By a moment Sir *Isaac Newton* understands a momentaneous increment, or decrement, of a flowing quantity, proportional to the velocity of the flowing quantity.

Here it is to be carefully observed, that, as for the absolute quantity or magnitude either of a moment, or of the time that a moment is generated in, Sir *Isac* makes no enquiry about it. He only gives us to understand, that it is less than any finite quantity. All that he is concerned about, is finding the first proportion of the nascent moments, or the proportion with which the moments begin to exist. And nascent moments being in the same proportion, as the velocities with which they begin to exist, *i. e.* in the same proportion, as the velocities of their flowing quantities; he teaches us, that it will answer the same end, if instead of the moments themselves, we make use of their velocities, or of any finite quantities in the same proportion with those velocities.

Accordingly, in explaining the sense of his *Lemma*, he puts the letters a, b, c to signify either the moments, or the velocities, of the flowing quantities A, B, C. And this it behoves us to take particular notice of, because these letters a, b, c do in the former case represent nascent quantities, and in the latter stand for finite magnitudes proportional to their respective velocities. Of which remark we shall see the use by and by

There is likewise a third acceptation, in which we may take the quantities represented by a, b, c. For these letters may be understood to represent the differences of *Monsieur Leibnitz*, as is plainly declared by the words subjoined to the *Scholium* of this *Lemma* in the first and second edition of the *Principia.* Where, after speaking of his own method of Fluxions, and of the method of Differences, which at that time Sir *Isaac* suffered to pass for the invention of *Monsieur Leibnitz*, he has these express words, *Utriusque* (*sc. methodi*) *fundamentum continetur in hoc Lemmate.*

This being premised, I come now to consider the demonstration of this *Lemma*; which with singular art is so worded, as to suit these three different significations of the letters a, b, and c.

If they stand either for nascent moments, or for finite quantities representing velocities; the demonstration, thoughout all the cases, will be perfectly just and accurate to the utmost rigor.

If they stand for differences; the demonstration in the first and most simple case will be perfectly just, as before. In the other more complex cases, it will not indeed be perfect, because something must be rejected, as infinitely small. But then, what is so rejected, will be infinitely smaller than that infinitely small quantity rejected by *Monsieur Leibnitz.* The reader will see, that I speak here *stylo differentiali.*

In order to make good these assertions, I proposed to consider the demonstration in every one of these different lights. And perhaps it may be of use, in order to see more clearly the design and contrivance of it, if, in the first place, we examine how, and how far it may be applied to the differential method of *Monsieur Leibnitz.* But before this can be done, we must a little consider what is meant by the word difference.

Monsieur Leibnitz, though he transiently speaks of his differences, as indivisible increments of variable quantities, as infinitely small, or less than any given quantity; yet when, a year or two after the publication of Sir *Isaac Newton*'s *Principia*, from many passages of which, it is probable, he might take occasion to reconsider his own principles, he comes particularly to explain himself about their magnitude, thinks he cannot more clearly express himself, than by calling them * incomparably small. And to illustrate his meaning, he instances the diameter of the earth, as being a line incomparably small in regard of the extent of the heavens. From which it is plain, that Gentleman was not altogether so bold, as some of his Followers have since been, in supposing infinite and infinitely small quantities, as fixed, determinate magnitudes actually existing. On the contrary, every one sees from this instance, that his differences were only finite quantities exceedingly small. At least, as Sir *Isaac Newton* never admitted of indivisibles, nor of quantities infinitely small, conceived as actually existing in a fixed, determinate and invariable state; it is plain that, whatever notion *Monsieur Leibnitz* might have entertained of his differences, yet Sir *Isaac*, in making them the subject of a geometrical demonstration, could look upon them as no other than finite quantities exceeding small, such as he himself calls *quam minimas*.

These differences therefore being regarded by Sir *Isaac Newton*, if not by *Monsieur Leibnitz* himself, as no other than exceedingly small finite quantities, we come now to see how this demonstration is to be understood, when applied to the differential method, beginning with the first and simplest case of that method, and proceeding to the more complex.

If the variable lines A and B, increasing uniformly, *i. e.* each of them receiving equal augmentations in equal portions of time, have for their differences, the quantities a and b: the rectangle AB, formed by those lines A and B, will have for its contemporary difference, the quantity $a\mathrm{B} + b\mathrm{A}$.

In order to demonstrate this, we may suppose,

1. That the differences a and b are thus uniformly generated in a given time.

2. That the sides of the variable rectangle attain their respective magnitudes A and B, exactly at the middle instant of that given time.

From which two suppositions it follows, that, as half the given time is elapsed, before the instant when the sides of the rectangle arrive at the magnitudes A and B; so likewise half of the differences a and b are generated before that same instant: And that in the other half of the given time, which passes after the sides are arrived at the magnitudes A and B, the remaining halves of the differences a and b are generated.

Therefore, at the beginning of the given time, the sides were of the magnitudes $\mathrm{A} - \frac{1}{2}a$ and $\mathrm{B} - \frac{1}{2}b$: and the rectangle was of the magnitude $\overline{\mathrm{A} - \frac{1}{2}a} \times \overline{\mathrm{B} - \frac{1}{2}b}$.

And at the end of the given time, the sides are of the magnitudes $\mathrm{A} + \frac{1}{2}a$ and $\mathrm{B} + \frac{1}{2}b$: and the rectangle is of the magnitude $\overline{\mathrm{A} + \frac{1}{2}a} \times \overline{\mathrm{B} + \frac{1}{2}b}$.

And as by subducting the magnitudes of the sides at the beginning of the given time, *viz.* $\mathrm{A} - \frac{1}{2}a$ and $\mathrm{B} - \frac{1}{2}b$, from the magnitudes of the sides at the end of the given time, *viz.* $\mathrm{A} + \frac{1}{2}a$ and $\mathrm{B} + \frac{1}{2}b$, we obtain a and b the differences of the sides generated during that given time:

So, by subducting the magnitude of the rectangle at the beginning of the given time, *viz.* $\overline{\mathrm{A} - \frac{1}{2}a} \times \overline{\mathrm{B} - \frac{1}{2}b}$, from the magnitude of the rectangle at the end of the given time, *viz.* $\overline{\mathrm{A} + \frac{1}{2}a} \times \overline{\mathrm{B} + \frac{1}{2}b}$, we shall obtain $a\mathrm{B} + b\mathrm{A}$, the difference of the rectangle, generated during

* Act. Lipsiens. 1689. p. 85.

the same given time.

This proceeding therefore of Sir *Isaac Newton*, in supposing one half of the differences to be generated, before the instant of time when the rectangle is of the magnitude AB, and one half after that instant, gives $a\mathrm{B} + b\mathrm{A}$, the difference of the rectangle, absolutely just and exact, without rejecting any part of that difference, as inconsiderable, on account of its smallness. And this it does, whether the differences a and b are supposed to be exceedingly small, or of any finite magnitude.

But *Monsieur Leibnitz*, in supposing the whole difference to be generated, after the instant of time when the rectangle is of the magnitude AB, finds $a\mathrm{B} + b\mathrm{A} + ab$ for the difference of the rectangle; and is obliged afterwards to reject the quantity ab, as being insignificant, or incomparably small with regard to the other terms $a\mathrm{B} + b\mathrm{A}$. His method therefore plainly appears to be less exact than Sir *Isaac Newton*'s, in this first and simplest case of the differential method.

And this same observation will easily be found to hold good in a second case, which, though somewhat more complex than the former, yet equally falls within the sense of the demonstration we have been considering. That is, when the sides A and B do not flow uniformly, but yet flow proportionably, *i. e.* in such manner, as that the first half of the difference of A is generated in the same time, as the first half of the difference B is generated; and the latter halves of those differences are likewise generated in the same time with each other; although the first time be not equal to the second time, or the whole time of generating the whole differences be not bisected, at the instant that the sides attain the magnitudes A and B.

It must be owned indeed, that, in the more complex cases of rectangles, when the sides vary neither uniformly, nor yet proportionably, or, which amounts to the same thing, in the cases of more compound quantities, whose sides flow uniformly, or proportionably, as in *Cas.* 2. of Sir *Isaac*'s demonstration, this contrivance of fixing an instant of time, before and after which one half of the difference of each side shall be generated, is not sufficient to obtain, upon the principles of *Monsieur Leibnitz*, a difference perfectly exact. Still there will be something to reject as incomparably small. But then this quantity rejected as incomparably small, will be incomparably smaller than that rejected by *Monsieur Leibnitz*. Of this it may be proper to give a few instances.

In the variable quantity ABC, whose sides flow uniformly, or proportionably, the difference taken, according to *Monsieur Leibnitz*'s method, is $a\mathrm{BC} + b\mathrm{AC} + c\mathrm{AB} + ab\mathrm{C} + ac\mathrm{B} + bc\mathrm{A} + abc$: out of which are rejected the terms $ab\mathrm{C} + ac\mathrm{B} + bc\mathrm{A} + acb$.

The difference of the same quantity taken by this way of Sir *Isaac Newton*, is $a\mathrm{BC}+b\mathrm{AC}+c\mathrm{AB} + \frac{1}{4}abc$; out of which is to be rejected only the last term $\frac{1}{4}abc$, which is incomparably small in respect of what is rejected by *Monsieur Leibnitz*.

In the cube A^3, the difference found by Monsieur Leibnitz's method is $3\mathrm{A}^2a+3\mathrm{A}a^2+a^3$, and the two last terms are rejected.

But the difference found by this course of proceeding of Sir *Isaac Newton*, is only $3\mathrm{A}^2a+\frac{1}{4}a^3$; and only the last term is rejected, which is incomparably small with regard to $3\mathrm{A}a^2+a^3$, the terms rejected by *Monsieur Leibnitz*.

In A^4 *Monsieur Leibnitz*'s difference is $4\mathrm{A}^3a + 6\mathrm{A}^2a^2 + 4\mathrm{A}a^3 + a^4$; and the three last terms are rejected.

The difference to be found in following Sir *Isaac Newton*, is $4\mathrm{A}^3a + \mathrm{A}a^3$; and only the last term is rejected, which is incomparably small in respect of $6\mathrm{A}^2a^2 + 4\mathrm{A}a^3 + a^4$.

In like manner, it will be found, that in every power of A, the quantity rejected by *Monsieur Leibnitz* as incomparably small, always includes the second power of a, or a^2; whereas the quantity to be rejected in pursuing Sir *Isaac Newton*'s method, never includes any thing greater than the third power of a, or a^3. But a^3 is incomparably small in regard of a^2.

This course therefore taken by Sir *Isaac Newton*, to find the difference of variable quantities, though not rigorously geometrical in the higher cases, yet approaches nearer to geometrick rigor than the method used by *Monsieur Leibnitz*; which is what I had given the *Author of the Analyst* to understand in my first Letter, *pag.* 50. But possibly it will be thought, I have spent more time about this matter, than it deserves. Let us therefore proceed to consider this demonstration, as relating to Sir *Isaac*'s own method.

It is evident that in *Cas.* 1. Sir *Isaac Newton* supposes one half of each moment, a and b, of the variable sides, A and B, of the rectangle, AB, to be generated before the instant of time, when those sides arrive at their respective magnitudes A and B; and the other half of each moment to be generated after that instant. From which supposition he plainly demonstrates, by the steps we have already considered, that, while the sides of the rectangle generate their respective moments a and b, the rectangle it self generates its contemporary moment $a\mathrm{B}+b\mathrm{A}$.

In this there can be no difficulty, while the sides of the rectangle flow either uniformly, or proportionably; in one of which senses Sir *Isaac*'s *Cas.* 1. is naturally to be understood.

But in *Case* 2, where the moment of ABC is determined, a little difficulty may occur to those, who have not the doctrine of prime and ultimate ratio's in their thoughts. It is, that if A, B and C are supposed to flow uniformly, or proportionably, then AB, or G, cannot flow either uniformly, or proportionably, to C; since it must flow with a velocity, at least, uniformly accelerated. Consequently, when C wanted half its moment, $\frac{1}{2}c$, G could not want half its moment $\frac{1}{2}g$, or $\frac{1}{2}a\mathrm{B} + \frac{1}{2}b\mathrm{A}$, as it ought to do, in order to make this case come within *Cas.* 1; since it must really have wanted, not more than the quantity $\frac{1}{2}a\mathrm{B} + \frac{1}{2}b\mathrm{A} - \frac{1}{4}ab$. Also, when C has gained the other half of its moment $\frac{1}{2}c$, G will have gained more than half its moment, $\frac{1}{2}g$, since it will really have gained, at least, the quantity $\frac{1}{2}a\mathrm{B} + \frac{1}{2}b\mathrm{A} + \frac{1}{4}ab$.

But this objection is easily removed, by considering, that the two quantities, $\frac{1}{2}a\mathrm{B} + \frac{1}{2}b\mathrm{A} - \frac{1}{4}ab$ & $\frac{1}{2}a\mathrm{B} + \frac{1}{2}b\mathrm{A} + \frac{1}{4}ab$, do really come to an equality with each other, when a and b are diminished *ad infinitum*, by *Lemma* 1. *Libr.* 1. *Princip.* which is what I had asserted in my first *Letter*, p. 53, and is agreeable to the demonstration given in *p.* 85, 86. of my second *Letter* to the *Author of the Analyst.*

We come now to consider the third sense, in which this demonstration may be understood, by taking a and b for finite lines, proportionable to the velocities of the flowing sides A and B. And here if we suppose,

1. That the lines a and b, are portions of the sides A and B, and are generated uniformly in some finite time.

2. That the sides of the rectangle attain the magnitudes A and B, exactly at the middle instant of that finite time:

We may, by pursuing the Steps of the demonstration, obtain the quantity $a\mathrm{B} + b\mathrm{A}$, which will be the augmentation of the rectangle, generated in the same time with a and b, the augmentations of the sides.

It remains therefore only to shew, that this augmentation $a\mathrm{B} + b\mathrm{A}$, is proportional to the velocity of the rectangle AB.

And here it is first to be observed, that, although this quantity $a\mathrm{B} + b\mathrm{A}$ is generated in the same time with the lines a and b, yet it is not generated, as those lines, with an uniform velocity, but with a velocity uniformly accelerated during that time.

But then it is well known, that the same quantity will be generated in the same time, either by a velocity uniformly accelerated during that time, or by the velocity taken at the middle instant of that time, and uniformly continued during the whole time.

Therefore by the velocity of the rectangle, taken at the middle instant of the given time, *i. e.* by the velocity of the rectangle AB, if it were continued uniformly, the quantity $a\mathrm{B}+b\mathrm{A}$, would be generated in the same time, as the lines a and b are uniformly generated: Therefore the quantities $a\mathrm{B}+b\mathrm{A}$, $a\times 1$ and $b\times 1$, since they might be uniformly generated, in the same time, by the velocities of their respective flowing quantities AB, $\mathrm{A} \times 1$, and $\mathrm{B} \times 1$, must be proportional to those velocities.

We have therefore shown, that in this easiest and simplest case of a flowing rectangle, when the sides flow uniformly, the proportion between the velocities of the sides, and the velocity of the rectangle, is truly determined in the demonstration of *Cas.* 1. by Sir *Isaac Newton.*

And if we take a case more complicated, as that of GC in *Cas.* 2. of this demonstration, or any other case of a rectangle, whose sides flow with velocities unequally accelerated, or retarded, in any manner whatsoever, it will be easy to reduce such a case to that we have been considering, of a rectangle whose sides flow with uniform velocities.

For let CD be a variable rectangle, whose sides C and D flow with velocities howsoever accelerated or retarded: And at a given instant of time, when the sides attain the exact magnitudes C and D, let their present velocities be in the proportion of the finite lines c and d.

Also let AB be another variable rectangle, whose sides A and B, at the same given instant, are respectively equal to the sides C and D, but flow with uniform velocities respectively equal to the present velocities of C and D, and which may consequently be expressed by the finite lines a and b, respectively equal to the finite lines c and d.

It is manifest, that since the sides C and D, A and B are respectively equal in each rectangle, and their velocities are likewise respectively equal, the rectangles themelves CD, AB, are equal to each other, and their velocities are likewise equal.

But the finite quantities $a \times 1$, $b \times 1$, and $a\mathrm{B} + b\mathrm{A}$, are proportional to the velocities of $\mathrm{A} \times 1$, $\mathrm{B} \times 1$, and AB, respectively.

Consequently the respectively equal finite quantities $c \times 1$, $d \times 1$, and $c\mathrm{D} + d\mathrm{C}$, are proportional to the respectively equal velocities of $\mathrm{C} \times 1$, $\mathrm{D} \times 1$, and CD.

From all which it appears, that Sir *Isaac Newton*'s demonstration holds good, when finite moments, *i. e.* finite quantities proportional to the velocities, are substituted in the room of evanescent moments, as I acquainted the *Author of the Analyst* in *p.* 74. of my second *Letter.*