# ADDITIONAL APPLICATIONS OF THE THEORY OF ALGEBRAIC QUATERNIONS

**By**

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# *Additional Applications of the Theory of Algebraic Quaternions.*

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The following is the communication made by Sir William R. Hamilton on some additional applications of his theory of algebraic quaternions.

It had been shown to the Academy, at one of their meetings* in the last summer, that the differential equations of motion of a system of bodies attracting each other according to Newton's law, might be expressed by the formula:

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \Sigma \frac{m + \Delta m}{-\Delta\alpha\sqrt{(-\Delta\alpha^2)}} \tag{1}$$

in which $\alpha$ is the *vector* of any one such body, or of any elementary portion of a body, regarded as a material point, and referred to an arbitrary origin; $m$ the constant called its mass; $\alpha + \Delta\alpha$, and $m + \Delta m$, the vector and the mass of another point or body of the system; $\Sigma$ the sign of summation, relatively to all such other bodies, or attracting elements of the system; and d the characteristic of differentiation, performed with respect to $t$ the time.

If we confine ourselves for a moment to the consideration of *two* bodies, $m$ and $m'$, and suppose $r$ to be the positive number denoting the variable distance between them, so that $r$ is the *length* of the vector $\alpha' - \alpha$, and, therefore, by the principles of this calculus,

$$r = \sqrt{\{-(\alpha' - \alpha)^2\}}; \tag{2}$$

we shall have, by the formula (1), the two equations,

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{m'r^{-1}}{\alpha - \alpha'}, \quad \frac{\mathrm{d}^2\alpha'}{\mathrm{d}t^2} = \frac{m'r^{-1}}{\alpha' - \alpha};$$

which may also be thus written,

$$m\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = mm'r^{-3}(\alpha' - \alpha), \quad m'\frac{\mathrm{d}^2\alpha'}{\mathrm{d}t^2} = mm'r^{-3}(\alpha - \alpha'),$$

and which give

$$0 = m\left(\delta\alpha\,\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}\,\delta\alpha\right) + m'\left(\delta\alpha'\,\frac{\mathrm{d}^2\alpha'}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\alpha'}{\mathrm{d}t^2}\,\delta\alpha'\right) = 2\delta\frac{mm'}{r},$$

* See Appendix No III., page xxxvii. [Proceedings of the Royal Irish Academy, volume 3.]

$\delta\alpha$, $\delta\alpha'$ being any arbitrary infinitesimal variations of the vectors $\alpha$, $\alpha'$, and $\delta r$ being the corresponding variation of $r$; because

$$\begin{aligned}\delta\alpha\,(\alpha'-\alpha)+(\alpha'-\alpha)\,\delta\alpha+\delta\alpha'\,(\alpha-\alpha')+(\alpha-\alpha')\,\delta\alpha'\\ &= -(\delta\alpha'-\delta\alpha)(\alpha'-\alpha)-(\alpha'-\alpha)(\delta\alpha'-\delta\alpha)\\ &= -\delta\,.\,(\alpha'-\alpha)^2=\delta\,.\,r^2=2r\,\delta r=-2r^3\delta\,.\,r^{-1}.\end{aligned}$$

And by extending this reasoning to any system of bodies, we deduce from the equation (1) this other formula, by which it may be replaced:

$$\tfrac{1}{2}\Sigma\,.\,m\left(\delta\alpha\,\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}\,\delta\alpha\right)+\delta\Sigma\frac{mm'}{r}=0. \tag{3}$$

Although it is believed that this result (3), if regarded merely *as a symbolic form*, is new, as well as the method by which it has been here obtained; yet if we transform it by the introduction of rectangular coordinates, $x$, $y$, $z$, making for this purpose

$$\alpha=ix+jy+kz,\quad \alpha'=ix'+jy'+kz', \tag{4}$$

and eliminating the squares and products of the three imaginary units, $i$, $j$, $k$, by the nine fundamental relations which were communicated to the Academy in 1843, namely,

$$\left.\begin{array}{l} i^2=j^2=k^2=-1;\\ ij=k,\quad jk=i,\quad ki=j;\\ ji=-k;\quad kj=-i,\quad ik=-j;\end{array}\right\} \tag{5}$$

we are conducted, from the equation (3), to a well-known equation, of Lagrange, which may be written thus:

$$\Sigma\,.\,m\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\delta x+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\,\delta y+\frac{\mathrm{d}^2z}{\mathrm{d}t^2}\,\delta z\right)=\delta\Sigma\,.\,\frac{mm'}{r};$$

where $r$, by (2), (4), (5), is equal to the known expression,

$$r=\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}.$$

If the law of attraction were supposed different from that of the inverse square, a different function of $r$, instead of $r^{-1}$, should be multiplied by the product of the two masses.

But further, it is not difficult so to operate on the formula (3), as to deduce from it another equation which shall be equivalent to the forms that were proposed by the present author, in his papers "On a General Method in Dynamics" (published in the Philosophical Transactions),* as being, at least theoretically, forms for the *integrals* of the differential equations of motion of any system of attracting bodies. For if we observe, that by the

* 1834, Part II. 1835, Part I.

principles of the calculus of variations, combined with those of the method of vectors, we have the identity,

$$\delta\alpha\,\mathrm{d}^2\alpha + \mathrm{d}^2\alpha\,\delta\alpha = \mathrm{d}(\delta\alpha\,\mathrm{d}\alpha + \mathrm{d}\alpha\,\delta\alpha) - \delta(\mathrm{d}\alpha^2);$$

and if we write

$$v = \sqrt{\left\{-\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2\right\}}, \tag{6}$$

denoting by $v$ the magnitude or degree (but not the direction) of the velocity of the body of which the vector is $\alpha$; we may transform the equation (3) into the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma\,.\,\frac{m}{2}\left(\delta\alpha\,\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\,\delta\alpha\right) + \delta\left(\Sigma\frac{mv^2}{2} + \Sigma\frac{mm'}{r}\right) = 0; \tag{7}$$

which, when operated upon by the characteristic $\int_0^t \mathrm{d}t$, that is, when integrated once with respect to the time from 0 to $t$, becomes

$$\Sigma\,.\,\frac{m}{2}\left(\delta\alpha\,\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \delta\alpha_0\,\frac{\mathrm{d}\alpha_0}{\mathrm{d}t} + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\,\delta\alpha - \frac{\mathrm{d}\alpha_0}{\mathrm{d}t}\,\delta\alpha_0\right) + \delta\mathrm{F} = 0, \tag{8}$$

if we make for abridgment

$$\mathrm{F} = \int_0^t \mathrm{d}t\left(\Sigma\frac{mv^2}{2} + \Sigma\frac{mm'}{r}\right), \tag{9}$$

and denote by $\delta\alpha_0$ and $\frac{\mathrm{d}\alpha_0}{\mathrm{d}t}$ the values which the variation of $\alpha$, and the differential coefficient of that vector taken with respect to $t$, are supposed to have at the origin of time. The definite integral denoted here by the letter F is the same with that which was denoted by the letter S in the Essays already referred to, and which was called, in one of those Essays, the *Principal Function* of the motion of a system of bodies, and if we now regard it as a function of the time $t$, and of all the final and initial vectors $\alpha, \alpha' \ldots, \alpha_0, \alpha'_0 \ldots$ of the various bodies of the system, and suppose (as we say) that its variation, taken with respect to all those vectors, is determined by an equation of the form,

$$0 = 2\,\delta\mathrm{F} + \Sigma(\sigma\,\delta\alpha - \sigma_0\,\delta\alpha_0 + \delta\alpha\,.\,\sigma - \delta\alpha_0\,.\,\sigma_0), \tag{10}$$

in which $\sigma$, $\sigma_0$ are vectors, we are conducted, by comparison of the coefficients of the arbitrary variations of vectors, in the equations (8) and (10), to the two following systems of formulæ:

$$m\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \sigma, \quad m'\frac{\mathrm{d}\alpha'}{\mathrm{d}t} = \sigma', \quad \ldots; \tag{11}$$

$$m\frac{\mathrm{d}\alpha_0}{\mathrm{d}t} = \sigma_0, \quad m'\frac{\mathrm{d}\alpha'_0}{\mathrm{d}t} = \sigma'_0, \quad \ldots; \tag{12}$$

of which the former may be regarded as intermediate, and the latter as final integrals of the differential equations of motion. The determination of the (vector) coefficient $\sigma$, from the variation of the (scalar) function F, is an operation of the same kind as the known operation of taking a partial differential coefficient, and may, in these new calculations, be called by the same name; but in order to be fully understood, it requires some new considerations, of which the account must be postponed to another occasion.

Consider a system of *three* attracting masses $m$, $m'$, $m''$, with their corresponding vectors, $\alpha$, $\alpha'$, $\alpha''$; and make for abridgment $\alpha' - \alpha = \beta$, and $\alpha'' - \alpha = \gamma$; we shall have, by (1), for the differential equations of motion of these three masses, referred to an arbitrary origin of vectors, the following:

$$\left.\begin{aligned} \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} &= \frac{m'}{-\beta\sqrt{(-\beta^2)}} + \frac{m''}{-\gamma\sqrt{(-\gamma^2)}}; \\ \frac{\mathrm{d}^2(\alpha+\beta)}{\mathrm{d}t^2} &= \frac{m}{\beta\sqrt{(-\beta^2)}} + \frac{m''}{(\beta-\gamma)\sqrt{\{-(\beta-\gamma)^2\}}}; \\ \frac{\mathrm{d}^2(\alpha+\gamma)}{\mathrm{d}t^2} &= \frac{m}{\gamma\sqrt{(-\gamma^2)}} + \frac{m'}{(\gamma-\beta)\sqrt{\{-(\gamma-\beta)^2\}}}; \end{aligned}\right\} \tag{13}$$

which give, for the internal or relative motions of $m'$ and $m''$ about $m$, the equations:

$$\left.\begin{aligned} \frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} &= \frac{m+m'}{\beta\sqrt{(-\beta^2)}} + m''\left\{\frac{(\beta-\gamma)^{-1}}{\sqrt{\{-(\beta-\gamma)^2\}}} + \frac{\gamma^{-1}}{\sqrt{(-\gamma^2)}}\right\}; \\ \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} &= \frac{m+m''}{\gamma\sqrt{(-\gamma^2)}} + m'\left\{\frac{(\gamma-\beta)^{-1}}{\sqrt{\{-(\gamma-\beta)^2\}}} + \frac{\beta^{-1}}{\sqrt{(-\beta^2)}}\right\}. \end{aligned}\right\} \tag{14}$$

If we suppress the terms multiplied by $m''$ in the first of these equations (14), or the terms multiplied by $m'$ in the second of those equations, we get the differential equation of motion of a binary system, under a form, from which it was shewn to the Academy last summer, that the laws of Kepler can be deduced. If we take account of the terms thus suppressed, we have, at least in theory, the means of obtaining the perturbations.

Let $m$ be the earth, $m'$ the moon, $m''$ the sun; then $\beta$ and $\gamma$ will be the geocentric vectors of the moon and sun; and the laws of the disturbed motion of our satellite will be contained in the two equations (14), but especially in the first of these equations. By the principles of the present calculus we have the developments,

$$(\gamma-\beta)^{-1} = \gamma^{-1} + \gamma^{-1}\beta\gamma^{-1} + \gamma^{-1}\beta\gamma^{-1}\beta\gamma^{-1} + \dots, \tag{15}$$

and

$$\frac{\sqrt{(-\gamma^2)}}{\sqrt{\{-(\gamma-\beta)^2\}}} = \left\{1 - \frac{\beta\gamma+\gamma\beta}{\gamma^2} + \frac{\beta^2}{\gamma^2}\right\}^{-\frac{1}{2}} = 1 + \frac{\beta\gamma+\gamma\beta}{2\gamma^2} + \dots; \tag{16}$$

if then we reject the terms of the same order as $m''\beta^2\gamma^{-4}$, that is, terms depending on the inverse fourth power of the distance of the sun from the earth, the disturbing part of the expression for the second differential coefficient, taken with respect to the time, of the moon's geocentric vector, will reduce itself in this notation to the following:

$$\frac{-m''}{\sqrt{(-\gamma^2)}}\left(\frac{(\gamma-\beta)^{-1}\sqrt{(-\gamma^2)}}{\sqrt{\{-(\gamma-\beta)^2\}}}-\gamma^{-1}\right)=\tfrac{1}{2}m''(-\gamma^2)^{-\frac{3}{2}}(\beta+3\gamma^{-1}\beta\gamma). \tag{17}$$

This symbolic result admits of a simple geometrical interpretation. The symbol $\gamma^{-1}\beta\gamma$ denotes a vector in the plane of the two vectors $\beta$ and $\gamma$, which has the same length as $\beta$, and is inclined at the same angle to $\gamma$, but at the other side of that line; so that $\gamma$ bisects the angle between $\beta$ and $\gamma^{-1}\beta\gamma$. If then we conceive a fictitious moon among the stars, so situated that either the sun, or a point opposite to the sun, as seen from the earth, bisects the arc of a great circle on the celestial sphere, between the fictitious and the actual moon (the bodies being here treated as points); and if we decompose the sun's disturbing force on the moon into two others, directed respectively towards the extremities of that celestial arc which is in this manner bisected: one component force, resulting from this decomposition, will be constantly ablatitious, tending directly to increase the distance of the moon from the earth, and bearing to the attractive force in the moon's undisturbed relative orbit, a ratio compounded of the direct ratio of half the mass of the sun to the sum of the masses of the earth and moon, and of the inverse ratio of the cubes of the distances of the sun and moon from the earth; and *the other component force, directed towards the fictitious moon*, will be *exactly the triple of the ablatitious force* thus determined; provided that we still neglect all terms depending on the inverse fourth power of the sun's distance, as we have done in deducing the equation (17), of which the theorem here enunciated is an interpretation. A similar result, of course, hold good, for every satellite disturbed by the central body of a system. The theorem admits of being proved by considerations more elementary, but was suggested to the author by the analysis above described; which may be extended, by continuing the developments (15), (16), to the case of one planet disturbed by another, and to a more accurate theory of a satellite.

Without entering into any farther account at present of the attempts which he has made to apply the processes and notation of his calculus of quaternions, or method of vectors, to questions of physical astronomy, the author wished to state that he had found those processes, and that notation, adapt themselves with remarkable facility to questions and results respecting Poinsot's Theory of Mechanical Couples. A single *force*, of the ordinary kind, is naturally represented by a *vector*, because it is constructed or represented, in mathematical reasoning, by a straight line having direction; but also a *couple*, of the kind considered by Poinsot, is found, in Sir William Hamilton's analysis, to admit of being regarded as *the vector part of the product of two vectors*, namely, of those which represent respectively one of the two forces of the couple, and the straight line drawn to any point of its line of direction from any point on the line of direction of the other force. Composition of couples corresponds to addition of such vector parts; and the laws of equilibrium of several forces, applied to various points of a solid body, are thus included in the two equations,

$$\Sigma\beta=0; \quad \Sigma(\alpha\beta-\beta\alpha)=0; \tag{18}$$

the vector of the point of application being $\alpha$, and the vector representing the force applied at that point being $\beta$. The condition of the existence of a single resultant is expressed by the formula,

$$\Sigma\beta\,.\,\Sigma(\alpha\beta-\beta\alpha)+\Sigma(\alpha\beta-\beta\alpha)\,.\,\Sigma\beta=0. \tag{19}$$

Instead of the two equations of equilibrium (18), we may employ the single formula

$$\Sigma \,.\, \alpha\beta = -c \tag{20}$$

$c$ here denoting a scalar (or real) quantity, which is independent of the origin of vectors, and seems to have some title to be called the total *tension* of the system.

In mentioning finally some applications of his algebraic method to central surfaces of the second order, the author could not but feel that he spoke in the presence of persons, of whom several were much better acquainted with the general geometrical properties of those surfaces than he could pretend to be. But, while deeply conscious that he had much to learn in this department from his brethren of the Dublin School, as well as from mathematicians elsewhere, he ventured to hope that the novelty and simplicity of the symbolic forms which he was about to submit to their notice might induce some of them to regard the future development of the principles of his method as a task not unworthy of their co-operation. He finds, then, that if $\alpha$ and $\beta$ denote two arbitary but constant vectors, and if $\rho$ be a variable vector, the equation of an ellipsoid with the three arbitrary, and, in general, unequal axes, referred to the centre as the origin of vectors, may be put under the following form

$$(\alpha\rho + \rho\alpha)^2 - (\beta\rho - \rho\beta)^2 = 1. \tag{21}$$

One of its circumscribing cylinders of revolution is denoted by the equation

$$-(\beta\rho - \rho\beta)^2 = 1; \tag{22}$$

the plane of the ellipse of contact by

$$\alpha\rho + \rho\alpha = 0; \tag{23}$$

and the systems of the two tangent planes parallel hereto, by

$$(\alpha\rho + \rho\alpha)^2 = 1. \tag{24}$$

A hyperboloid of one sheet, touching the same cylinder in the same ellipse, is denoted by the equation

$$(\alpha\rho + \rho\alpha)^2 + (\beta\rho - \rho\beta)^2 = -1; \tag{25}$$

its asymptotic cone by

$$(\alpha\rho + \rho\alpha)^2 + (\beta\rho - \rho\beta)^2 = 0; \tag{26}$$

and a hyperboloid of two sheets, with the same asymptotic cone (26), and with the tangent planes (24), is represented by a the formula

$$(\alpha\rho + \rho\alpha)^2 + (\beta\rho - \rho\beta)^2 = 1. \tag{27}$$

By changing $\rho$ to $\rho - \gamma$, in which $\gamma$ is a third arbitrary but constant vector, we introduce an arbitrary origin of vectors, or an arbitary position of the centre of the surface as referred to such an origin; and the general problem of determining that individual surface of the second order (supposed to have a centre, until the calculation shall show in any particular question that it has none), which shall pass through *nine given points*, may thus be regarded as equivalent to the problem of finding *three constant vectors*, $\alpha$, $\beta$, $\gamma$, which shall for nine given values of the variable vector $\rho$, satisfy one equation of the form

$$\{\alpha(\rho-\gamma)+(\rho-\gamma)\alpha\}^2 \pm \{\beta(\rho-\gamma)-(\rho-\gamma)\beta^2\}^2 = \pm 1; \tag{28}$$

with suitable selections of the two ambiguous signs, depending on, and in their turn determining, the particular nature of the surface. It is not difficult to transform the equation (28), or those which it includes, so as to put in evidence some of the chief properties of surfaces of the second order, with respect to their circular sections.

The recent expressions may be abridged, if we agree to employ the letters S and V as characteristics of the operations of taking separately the scalar and vector parts of any quaternion to which they are prefixed; for then we shall have

$$\alpha\rho+\rho\alpha = 2\mathrm{S}\,.\,\alpha\rho, \quad \beta\rho-\rho\beta = 2\mathrm{V}\,.\,\beta\rho; \tag{29}$$

so that, by making for abridgment $2\alpha = \alpha'$, $2\beta = \beta'$, the equation (21) of the ellipsoid (for example) will take the shorter form,

$$(\mathrm{S}\,.\,\alpha'\rho)^2 - (\mathrm{V}\,.\,\beta'\rho)^2 = 1. \tag{30}$$

Another modification of the notation, which, from its general character, will often be found useful, or at least illustrative, may be obtained by agreeing to denote by the geometrical symbol BA the vector $\beta-\alpha$, which is the difference of two other vectors $\alpha$ and $\beta$ drawn to the two points A and B, from any common origin; so that BA is the vector *to* B *from* A. Denoting also by the symbol CBA the quaternion CB $\times$ BA, which is the product of the two vectors CB and BA; by DCBA the continued product DC $\times$ CB $\times$ BA, and so on: the foregoing equations of central surfaces may be transformed, and a great number of geometrical processes and results expressed under concise and not inelegant forms. For example, the symbols

$$\frac{\mathrm{V}\,.\,\mathrm{ABC}}{\mathrm{AC}}, \tag{31}$$

and

$$\frac{\mathrm{S}\,.\,\mathrm{ABCD}}{\mathrm{V}\,.\,\mathrm{ABC}}, \tag{32}$$

will denote, in length and direction, the perpendiculars let fall, respectively, from the summit B on the base AC of a triangle, and from the summit D on the base ABC of a tetrahedron: the sextuple area of this tetrahedron ABCD being expressed in the same notation by the symbol S . ABCD.

The developments (15) and (16), with a great number of others, may be included in a formula which corresponds to Taylor's theorem, namely, the following:

$$f(\alpha+\mathrm{d}\alpha) = \left(1+\frac{\mathrm{d}}{1}+\frac{\mathrm{d}^2}{1\,.\,2}+\dots\right)f\alpha; \tag{33}$$

the only new circumstance being, that in interpreting or transforming the separate terms, for example, the term $\frac{1}{2}\mathrm{d}^2 f\alpha$, of the resulting development of the function $f(\alpha + \mathrm{d}\alpha)$, if $\alpha$ and its differential $\mathrm{d}\alpha$ denote vectors, we must in general employ *new rules of differentiation*, having indeed a close affinity to the known rules, but modified by the non-commutative character of the operation of multiplication in this calculus of vectors or of quaternions. It is thus that, instead of writing $\mathrm{d} \,.\, \alpha^2 = 2\alpha\,\mathrm{d}\alpha$, $\delta \,.\, \alpha^2 = 2\alpha\,\delta\alpha$, we have been obliged to write

$$\mathrm{d} \,.\, \alpha^2 = \alpha \,.\, \mathrm{d}\alpha + \mathrm{d}\alpha \,.\, \alpha; \tag{34}$$

$$\delta \,.\, \alpha^2 = \alpha \,.\, \delta\alpha + \delta\alpha \,.\, \alpha. \tag{35}$$