

**ILLUSTRATIONS FROM GEOMETRY OF THE
THEORY OF ALGEBRAIC QUATERNIONS**

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(Proceedings of the Royal Irish Academy, 3 (1847), Appendix, pp. xxxi–xxxvi.)

Edited by David R. Wilkins

2000

Illustrations from Geometry of the Theory of Algebraic Quaternions.

By Sir WILLIAM R. HAMILTON.

Communicated February 10, 1845.

[*Proceedings of the Royal Irish Academy*, vol. 3 (1847), Appendix, pp. xxxi–xxxvi.]

The spirit of Sir William Hamilton's communication, which was designed as a further illustration from geometry of the author's theory of algebraic quaternions, consisted in regarding operations on such quaternions as admitting of being ultimately interpreted as *operations on straight lines*; each line being considered as having not only a determinate *length*, but also a determinate *direction*. The *quotient* $\frac{b}{a}$ obtained by the division of one such line (b) by another (a), is, generally, in the author's view, a *quaternion*; it depends, in general, on *four* distinct elements, of which one, namely, the *modulus*, is a positive or absolute number expressing the *relative magnitude* of the dividend and divisor lines, while the three other elements serve jointly to express the *relative direction* of those two lines. Of the three latter, one is the *amplitude*, and marks the *inclination* of one line to the other, or the magnitude of the angle which they include; while the two others determine the *plane* of that angle, and are what have been called, in a former communication, the *directional coordinate*, such as the *longitude* and *colatitude* of the quaternion. In this comparatively geometrical view, as in the more algebraical view which was formerly stated to the Academy, the consideration of these four elements, modulus, amplitude, longitude, and colatitude, presents itself, therefore, naturally. We may also speak of the *axis* of a quaternion, meaning thereby the axis perpendicular to the plane of the two straight lines of which that quaternion is a quotient; and may say, that such an axis is itself *positive* or *negative*, or that it is taken in the positive or in the negative direction, according as it is the axis of a positive or a negative *rotation*, from the divisor to the dividend line. Quaternions may be said to be *coaxal* when their axes coincide, or only differ in sign. A quaternion is not altered in value when the two lines of which it is the quotient are transferred, without altering their directions, to any other positions in space; or when their lengths are both changed together in any common ratio; or when they are both made to revolve together, through any common amount of rotation, round the axis of the quaternion, without ceasing to be still in (or parallel to) the same common plane as before. It is, therefore, always possible to prepare any two proposed quaternions, or geometrical quotients or fractions of the kind above described, so as to have one *common denominator* or divisor line; and then the *addition* or *subtraction* of those two quaternions is effected, by retaining that common line as the denominator or divisor of the new quaternion, and by adding or subtracting the numerator lines, in order to obtain the new numerator of the same new quaternion, that is to say, of the sum or difference of the two old quaternions; addition and subtraction of straight *lines* (when those lines are supposed to have not only lengths but also *directions*) being performed according to the rules which have already been proposed

by several writers, and which correspond to compositions and decompositions of rectilinear *motions* (or of forces). *Multiplication* of two quaternions may be effected by preparing them so that the denominator (b) of the multiplier, may be equal to, or the same line with, the numerator (b) of the multiplicand (lines being *equal* when their directions as well as their lengths are the same), and by then treating the numerator (c) of the multiplier as the numerator of the product, and the denominator (a) of the multiplicand as the denominator of the product: and *division* may be regarded as the return to the multiplier, from a given product and multiplicand.

With this view of multiplication, it is evident that the product of the moduli of the two factors is equal to the modulus of the product. It is clear also, that if we construct a spherical triangle ABC, of which the three corners, or the radii drawn to them from the centre of the sphere, represent the directions of the three lines a, b, c, then the arc, or side of the triangle, AB, will represent the amplitude of the multiplicand quaternion; the arc or side BC will represent the amplitude of the multiplier; and the remaining arc or side AC the amplitude of the product, so that the spherical triangle will be constructed with these three amplitudes for its three sides. And we see that in the triangle thus constructed, the spherical angles at A and C, which are respectively opposite to the amplitudes of the multiplier and multiplicand, are equal to the respective inclinations of the axes of the multiplicand and multiplier to the axis of the product of the quaternions; and that the remaining spherical angle at B, which is opposite to the amplitude of the product, is equal to the supplement of the inclination of the axes of the factors to each other: a form almost the same with that under which the fundamental connexion of quaternions with spherical trigonometry was stated by Sir William Hamilton, in his first letter on the subject, to John T. Graves, Esq., which was written in October 1843, and has been printed in the Supplementary number of the *Philosophical Magazine* for December 1844. The other form of the same fundamental connexion, which was communicated to the Academy in November, 1843, may be deduced from the foregoing, by the consideration of that polar or supplementary triangle, of which the corners mark the directions of the *axes* of the factors and the product, and were then called the *representative* points of the three quaternions compared. If the order of the factors be changed, the (positive) axis of the product falls to the other side of the plane of the axes of the factors, being always so situated that the rotation round the axis of the multiplier from the axis of the multiplicand to that of the product is positive; *multiplication of quaternions* is therefore seen, in this as in other ways, to be *not in general a commutative operation*, or the result depends, in general, essentially on the *order* in which the factors are taken.

The same remarkable conclusion follows from the comparison of the lately mentioned spherical triangle ABC with another triangle C'BA', vertically opposite and equal thereto, and such that the common corner B bisects each of the two arcs C'C, A'A, joining the two pairs of corresponding corners; which other triangle may represent the directions of three lines c', b, a', related to the system of the three former lines c, b, a, by the following equations between geometrical quotients, or quaternions,

$$\frac{a'}{b} = \frac{b}{a}, \quad \frac{b'}{c} = \frac{c}{b};$$

for then, by the definition of multiplication of such quotients here proposed, we have the two

different results,

$$\frac{c}{b} \times \frac{b}{a} = \frac{c}{a}; \quad \frac{b}{a} \times \frac{c}{b} = \frac{a'}{c'};$$

and although these two resulting quaternion products have equal moduli and equal amplitudes, yet they have in general *different axes*, because the arcs AC and A'C', though equally long, are parts of different great circles, and are therefore situated in different planes. However, in that particular but useful and often occurring case, where the two factors have one common axis, the order of those factors becomes indifferent; and if attention be paid to positive and negative signs, it may be said that *coaxial quaternions* may be multiplied together, in either order, by adding their amplitudes, multiplying their moduli, and retaining their common axis. In general, it may be proved, from the views here given of multiplication and addition, that, although the *commutative* property of ordinary multiplication does not usually extend to operations on quaternions, yet the *distributive* and *associative* properties of that operation do always so extend; and that the commutative and associative properties of addition hold good in like manner for quaternions: results which were indeed stated to the Academy in November 1843, as consequences from the algebraical definitions of a quaternion, and of operations performed thereon, but have now been mentioned again, as following from more geometrical definitions also.

Comparing the view here proposed with that which was submitted to the Academy in November 1844, a quaternion may be said to reduce itself to a *scalar* (or ordinary real number), when the two straight lines, of which it is the quotient, are *parallel*; the scalar being positive when those lines are similar, but negative when they are opposite in direction. And, on the other hand, the scalar part vanishes, and the quaternion becomes a pure *vector*, when it is a quotient of two *rectangular* lines: and, in this last case, it may be conveniently constructed by a *third line* perpendicular to both of them, namely, by one drawn in the direction of the positive axis of the quaternion, with a length which bears to an assumed unit of length the ratio marked by the modulus. This third line, which thus represents or constructs the quotient of two other lines perpendicular to it and to each other, may, by a suitable choice of those two lines, receive any proposed length, and any proposed direction; and *every straight line having length and direction in space* may, in this view, be regarded as a particular quaternion, namely, as one of the class above called *vectors*. It is easy to *prove* that when lines are thus treated as quotients, they have the same sums, differences, and quotients, as those obtained by the processes or conceptions above described or alluded to; and hence it would be natural to *define*, as we should be at liberty to do, that the *product* of two lines is also in general a quaternion, obtained by multiplying two vector factors together, according to the rules of multiplication of quaternions. We should then be able to establish, in this new way, all the rules, already communicated to the Academy, for the *multiplication of straight lines in space*; and especially should be conducted anew to those *two rules*, or principles, which presented themselves to the author in his earliest researches on quaternions (as described in the printed letter already referred to), and which he still regards as fundamental in their theory: namely, first, that *the product of two straight lines, which agree in direction, is to be considered as a negative number*, namely, as the product of their two lengths taken negatively; and, secondly, that *the product of two rectangular lines is to be regarded as a third line perpendicular to both*, of which the length represents the product of their lengths, and to which the rotation, from the multiplicand line, round the multiplier line, is positive. The paradoxical, or, at

least, unusual appearance of these two fundamental rules, combined with the variety of the applications of which the author has found them susceptible, induce him to hope that he shall be pardoned for thus offering new confirmations or new illustrations of them, derived from considerations of the manner in which they present themselves from various points of view.