Kähler-Dirac Fermions and Exact Lattice Supersymmetry

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Lattice Supersymmetry – problems

Generic discretizations break SUSY completely

$\{Q, \bar{Q}\} = \gamma^\mu p_\mu$ – no p_μ!

Leads to fine tuning problem as $a \to 0$

Try to find formulations which preserve an element of SUSY

Two approaches currently on market:

- Orbifold supersymmetric matrix model (Kaplan, Unsal et al. – see Lattice 2003)
- Use twisted formulation of continuum SUSY theory.

Both ideas only work for extended SUSY $\mathcal{N} > 1$

Motivation

SYM models with extended (maximal) SUSY conjectured to be dual to gravitational theories eg.

- 4D $\mathcal{N} = 4$ $U(N)$ SYM dual to type IIB supergravity on $AdS_5 \times S^5$ in t'Hooft limit where $\lambda = g^2 N >> 1$ held fixed and $N \to \infty$.
- General conjecture: strongly coupled large N SYM theories in $p + 1$ dims dual to black p-brane solutions in supergravity.
- Transitions between different black hole solutions have an interpretation as thermal phase transitions in the Yang-Mills system.

Basic Idea

- Extended SUSY – *twist* to decompose fermions as set of p-form fields (reinterpret as Kähler-Dirac field)
  Consequence – expose scalar supercharge $\{Q, Q\} = 0$ and find action $S = Q\Lambda$
- If can maintain $Q^2 = 0$ for lattice fields can build discrete models **exactly** invariant under Q - reduce fine tuning
- Bonus – geometrical formulation of theory can be discretized without inducing *fermion doubling*

Recent work on twisted formulations

Catterall and Gregory, Phys. Lett. B487 (2000) 349.

Catterall and Karamov, Phys. Rev. D65 (2002) 094501

Giedt and Poppitz, JHEP 0405 (2004) 044

Giedt, Koniuk, Poppitz and Yavin, JHEP 0412 (2004) 033

Catterall, JHEP 0305 (2003) 038

Catterall and Ghadab, JHEP 0405 (2004) 044

Sugino, JHEP 0403 (2004) 067, JHEP 0501 (2005) 016.

Kato, Kawamoto and Uchida, Int. J. Mod. Phys. A19 (2004) 2149.

D'Adda, Kanamori, Kawamoto and Nagata, Nucl. Phys. B707 (2005) 100

Kato, Kawamoto and Miyake, hep-th/0502119

Catterall, JHEP 0411 (2004) 006, JHEP 06 027 (2005)

Giedt, hep-lat/0507016.

D'Adda, Kanamori, Kawamoto and Nagata, hep-lat/0507029.

Summary

- 2D Twist, Kähler-Dirac interpretation
- Lattice $\mathcal{N} = 2$ SYM in D=2
- Numerical results
- 4D Twist, Lattice $\mathcal{N} = 4$ SYM
- Conclusions/Future work

2D Twist

Consider theory with $\mathcal{N}=2$ supersymmetry (Euclidean space)

Two Majorana supercharges q^I_α with

$$\alpha \text{ -- SO(2) Lorentz index}$$
$$I \text{ -- SO(2) R-symmetry index}$$

Twist – new rotation group (Witten – topological FT)

$SO(2)'$=diagonal subgroup($SO(2)\times SO(2)_R$)

Equivalent to $I\to\beta$ and

$$q^I_\alpha \to q_{\alpha\beta}$$

Natural to expand on basis of 2D gamma matrices

$$q = QI + Q_\mu\gamma_\mu + Q_{12}\gamma_1\gamma_2$$

Original SUSY algebra implies twisted algebra

$$\{q,q\}_{\alpha\beta} = 4\gamma^\mu_{\alpha\beta}p^\mu$$

2D Twist continued

In components

$$\begin{aligned}\{Q,Q\} &= \{Q_{12},Q_{12}\} = \{Q,Q_{12}\} = \{Q_\mu,Q_\nu\} = 0\\ \{Q,Q_\mu\} &= P_\mu\\ \{Q_{12},Q_\mu\} &= -\epsilon_{\mu\nu}P_\nu\end{aligned}$$

Momentum P is Q-exact!

Plausible that $T_{\mu\nu}$ also then Q-exact.

Action is then Q-exact! $S = Q\Lambda$

Notice: to match 4 supercharges of SUSY theory take twisted supercharges to be real.

Implies reality condition on q:

$$\overline{q} = q^T \text{ in 2D}$$

2D Twisted Fields

If supercharges form matrix so do fermions

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12} \gamma_1 \gamma_2$$

Abstract p-form components and consider the fermions as represented by real Kähler-Dirac field (Kawamoto)

$$\Psi = (\frac{\eta}{2}, \psi_\mu, \chi_{12})$$

Original Dirac equation for 2 (Majorana) fermions equivalent to

$$(d - d^\dagger)\Psi = 0$$

corresponding to action

$$S_F = \frac{1}{2} \psi^\dagger_\mu \partial_\mu \eta + \frac{1}{2} \chi^\dagger_{\mu\nu} \partial_{[\mu} \psi_{\nu]}$$

True also when gauged $\partial \to D$

2D Twisted Fields continued

Fermions: $\Psi = (\frac{\eta}{2}, \psi_\mu, \chi_{12})$
Any Q-invariant theory must also contain superpartners
$\Phi = (\overline{\phi}, A_\mu, B_{12})$ with

$$Q^2\Phi = Q^2\Psi = 0 \quad \text{mod possible G.T}$$

Look like the fields of $\mathcal{N} = 2$ SYM in 2D!
(take in adjoint eg $\Phi \equiv \sum_a \Phi^a T^a$ with AH generators of $U(N)$)
Expect

$$S = Q\mathrm{Tr}\Lambda(\Psi, \Phi)$$

Twisted $\mathcal{N} = 2$ SYM in $D = 2$

Continuum twisted action:

$$\begin{aligned} S &= \beta Q \mathrm{Tr} \int d^2x \left(\frac{1}{4}\eta[\phi,\overline{\phi}] + 2\chi_{12}F_{12} + \right. \\ &+ \left. \chi_{12}B_{12} + \psi_\mu D_\mu \overline{\phi} \right) \end{aligned}$$

where $Q^2 =$ G.T parametrized by ϕ

$$\begin{aligned} QA_\mu &= \psi_\mu \\ Q\psi_\mu &= -D_\mu\phi \\ Q\phi &= 0 \\ Q\chi_{12} &= B_{12} \\ QB_{12} &= [\phi, \chi_{12}] \\ Q\overline{\phi} &= \eta \\ Q\eta &= [\phi, \overline{\phi}] \end{aligned}$$

Twisted SYM action continued

Vary, integrate out B_{12}:

$$\begin{aligned} S &= \beta \mathrm{Tr} \int d^2x \left(\frac{1}{4}[\phi, \overline{\phi}]^2 - \frac{1}{4}\eta[\phi, \eta] - F_{12}^2 \right. \\ &- D_\mu \phi D_\mu \overline{\phi} - \chi_{12}[\phi, \chi_{12}] \\ &- 2\chi_{12}\left(D_1\psi_2 - D_2\psi_1\right) - 2\psi_\mu D_\mu \eta/2 \\ &+ \left. \psi_\mu[\overline{\phi}, \psi_\mu] \right) \end{aligned}$$

Points to note:

1. Scalar+gauge part positive definite along contour $\overline{\phi}^a = (\phi^a)^*$ (AH generators) – recognise as bosonic sector of 2D $\mathcal{N} = 2$ SYM

2. Twisted fermion kinetic term = Kahler-Dirac action

Equivalence to spinor formulation

Use Euclidean chiral rep. for gamma matrices

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

define spinor λ as

$$\lambda = \begin{pmatrix} \frac{\eta}{2} - i\chi_{12} \\ \psi_1 - i\psi_2 \end{pmatrix}$$

Twisted fermion action = Dirac action

$$S = \lambda^\dagger M(\phi)\lambda$$

where

$$M = \gamma.D + \frac{(1+\gamma_5)}{2}[\overline{\phi}, \cdots] - \frac{(1-\gamma_5)}{2}[\phi, \cdots]$$

Lattice prescription (cubic)

- Map continuum scalars to fields on sites, 1-forms to links, 2-forms to plaquettes.
- 2 orientations ($p > 0$) – 2 independent lattice fields for each continuum field with non-zero spin. Represent using f and $f^\dagger$.
- Gauge transformations (2D):

$$\begin{aligned} f(x) &\rightarrow G(x)f_(x)G^{-1}(x) \\ f_\mu(x) &\rightarrow G(x)f_\mu(x)G^{-1}(x+\mu) \\ f_{12}(x) &\rightarrow G(x)f_{12}(x)G^{-1}(x+1+2) \end{aligned}$$

- $A_\mu(x) \rightarrow U_\mu(x) = e^{A_\mu(x)}$
  Complex A – U_μ and $U_\mu^\dagger$ independent.

Lattice derivatives

$$
\begin{aligned}
D^+_\mu f(x) &= U_\mu(x) f(x+\mu) - f(x) U_\mu(x) \\
D^+_\mu f_\nu(x) &= U_\mu(x) f_\nu(x+\mu) - f_\nu(x) U_\mu(x+\nu)
\end{aligned}
$$

Reduce to continuum derivatives as $a \to 0$ (and preserve GT props)

$$
\begin{aligned}
D^-_\mu f_\mu(x) &= f_\mu(x) U^\dagger_\mu(x) - U^\dagger_\mu(x-\mu) f_\mu(x-\mu) \\
D^-_\mu f_{\mu\nu}(x) &= f_{\mu\nu}(x) U^\dagger_\mu(x+\nu) - U^\dagger_\mu(x-\mu) f_{\mu\nu}(x-\mu)
\end{aligned}
$$

Avoid spectrum doubling (Rabin, Joos) if:

$$
\begin{gathered}
\partial_\mu \to D^+ \text{ if acts like } d \\
\partial_\mu \to D^- \text{ if acts like } d^\dagger
\end{gathered}
$$

Notice also:

$$
F_{\mu\nu}(x) = D^+_\mu U_\nu(x) \to F^{\text{cont}}_{\mu\nu} \text{ as } a \to 0
$$

Component lattice action

$$
\begin{aligned}
S_L &= \frac{1}{2}\beta Q \mathrm{Tr} \sum_x \left(-\frac{1}{4}\eta^\dagger [\phi, \overline{\phi}] - 2\chi_{12}^\dagger F_{12} \right. \\
&- \left. \chi_{12}^\dagger B_{12} - \psi_\mu^\dagger D_\mu^+ \overline{\phi} + \mathrm{h.c} \right)
\end{aligned}
$$

Notice lattice G.T properties require complex fields

$$
\begin{aligned}
QU_\mu &= \psi_\mu \\
Q\psi_\mu &= -D_\mu^+ \phi \\
Q\phi &= 0 \\
Q\chi_{12} &= B_{12} \\
QB_{12} &= [\phi, \chi_{12}]^{(12)} \\
Q\overline{\phi} &= \eta \\
Q\eta &= [\phi, \overline{\phi}]
\end{aligned}
$$

Lattice action continued

Carrying out Q-variation and int. B

$$
\begin{aligned}
S_L &= \frac{\beta}{2}\mathrm{Tr}\sum_x \Bigg(\frac{1}{4}[\phi,\overline{\phi}]^2 + F_{12}^\dagger F_{12} \\
&- \frac{1}{4}\eta^\dagger[\phi,\eta] - \chi_{12}^\dagger[\phi,\chi_{12}]^{(12)} + \psi_\mu^\dagger[\overline{\phi},\psi_\mu]^{(\mu)} \\
&+ (D_\mu^+\phi)^\dagger D_\mu^+\overline{\phi} - 2\chi_{12}^\dagger\left(D_1^+\psi_2 - D_2^+\psi_1\right) \\
&- 2\psi_\mu^\dagger D_\mu^+\frac{\eta}{2} + \mathrm{h.c}\Bigg)
\end{aligned}
$$

Invariant under Q, finite gauge transformations and $U(1)$

$$\psi_\mu \to e^{i\alpha}\psi_\mu,\quad \eta,\chi_{12} \to e^{-i\alpha}\eta,\chi_{12}$$

$$\overline{\phi} \to e^{-2i\alpha}\overline{\phi},\quad \phi \to e^{2i\alpha}\phi$$

Gauge action

$$\beta \mathrm{Tr} \sum_x F_{12}^\dagger(x) F_{12}(x)$$

$$\beta \mathrm{Tr} \sum_x \left(2I - U_P - U_P^\dagger\right) + \beta \mathrm{Tr} \sum_x (M_{12} + M_{21} - 2I)$$

where

$$U_P = U_1(x) U_2(x+1) U_1^\dagger(x+2) U_2^\dagger(x)$$

and

$$M_{12} = U_1(x) U_1^\dagger(x) U_2^\dagger(x+1) U_2(x+1)$$

Notice:

2nd term is zero if $U_\mu^\dagger(x) U_\mu(x) = I$.

Action collapses to usual Wilson action!

Twisted Fermions

$$\Psi = \begin{pmatrix} \eta/2 \\ \chi_{12} \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

Action $\Psi^\dagger M \Psi$

$$M = \begin{pmatrix} -[\phi,]^{(p)} & K \\ -K^\dagger & [\overline{\phi},]^{(p)} \end{pmatrix}$$

$$K = \begin{pmatrix} D_2^+ & -D_1^+ \\ -D_1^- & -D_2^- \end{pmatrix}$$

After integration – $\mathrm{Pf}(M)$.

In free limit $\mathrm{Pf}(M) = \det(K) = \det(D_\mu^+ D_\mu^-)$–no doubles!

Continuum Limit

Lattice formulation given in terms of complex fields.

Target continuum theory corresponds to setting

$$\begin{aligned} \mathrm{Im} X_\mu^a &= 0 \text{ all fields } X \text{ bar scalars} \\ \overline{\phi} &= -\phi^\dagger \end{aligned}$$

Question: are the Ward identities W.I corresponding to Q satisfied after this projection ?

Conjecture: yes (at least for large β)

Why?

Q-exact S allows W.I to be computed exactly for $\beta \to \infty$

Use $U_\mu(x) = R_\mu(x) u_\mu(x)$ with R hermitian, pos def. and u unitary: soln. of $F_{12}^\dagger F_{12} = 0$ is just $R_\mu(x) = 1$!

Q trans. imply all fields (bar scalars) can be taken real.

Simulations

Use unitary $U_\mu(x)$. Add mass term $m^2\phi^\dagger\phi$ to control I.R divergence of scalars

Bosonic action gauge inv. and real, pos. def. still

Fermion weight $\det^{\frac{1}{4}}(M^\dagger M + m^2)$ (neglect phase)

Use RHMC alg. to simulate exactly (Clark, Kennedy)

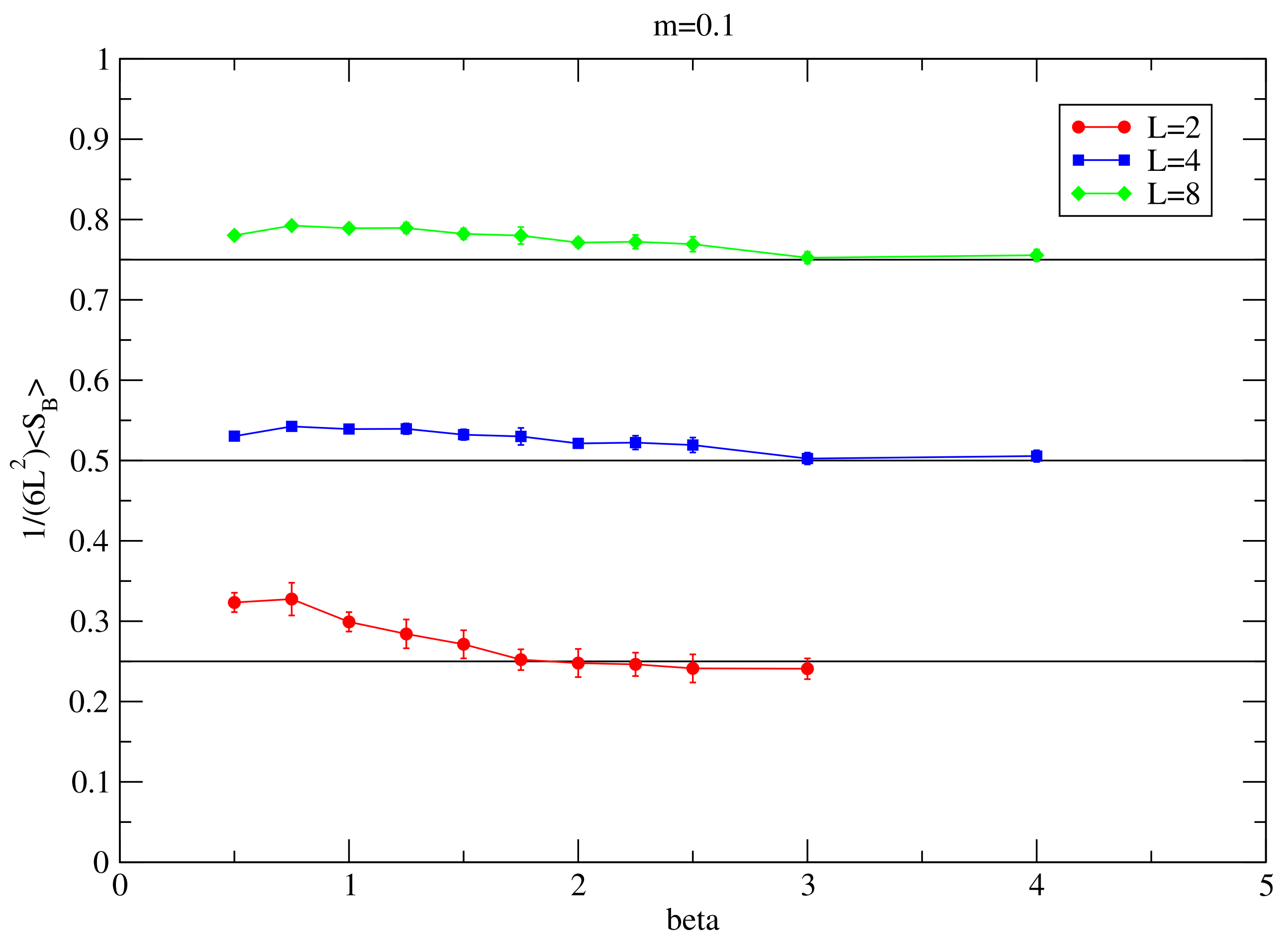
Parameters: $L = 2, 4, 8$, $m = 0.1, 0.05$, $\beta = 0.5 \to 4.0$

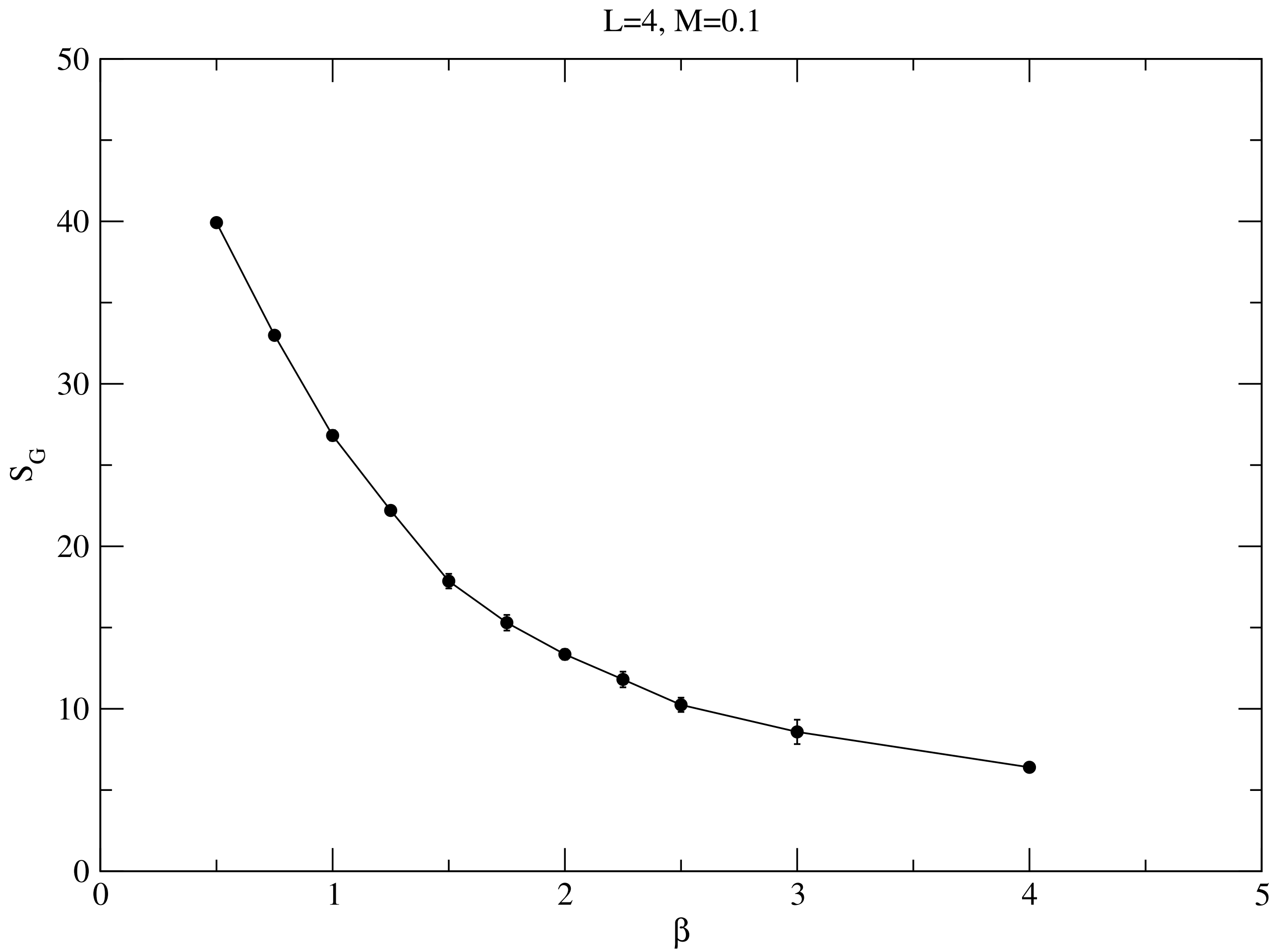
(notice: equiv. to $L = 4, 8, 16$ staggered lattices)

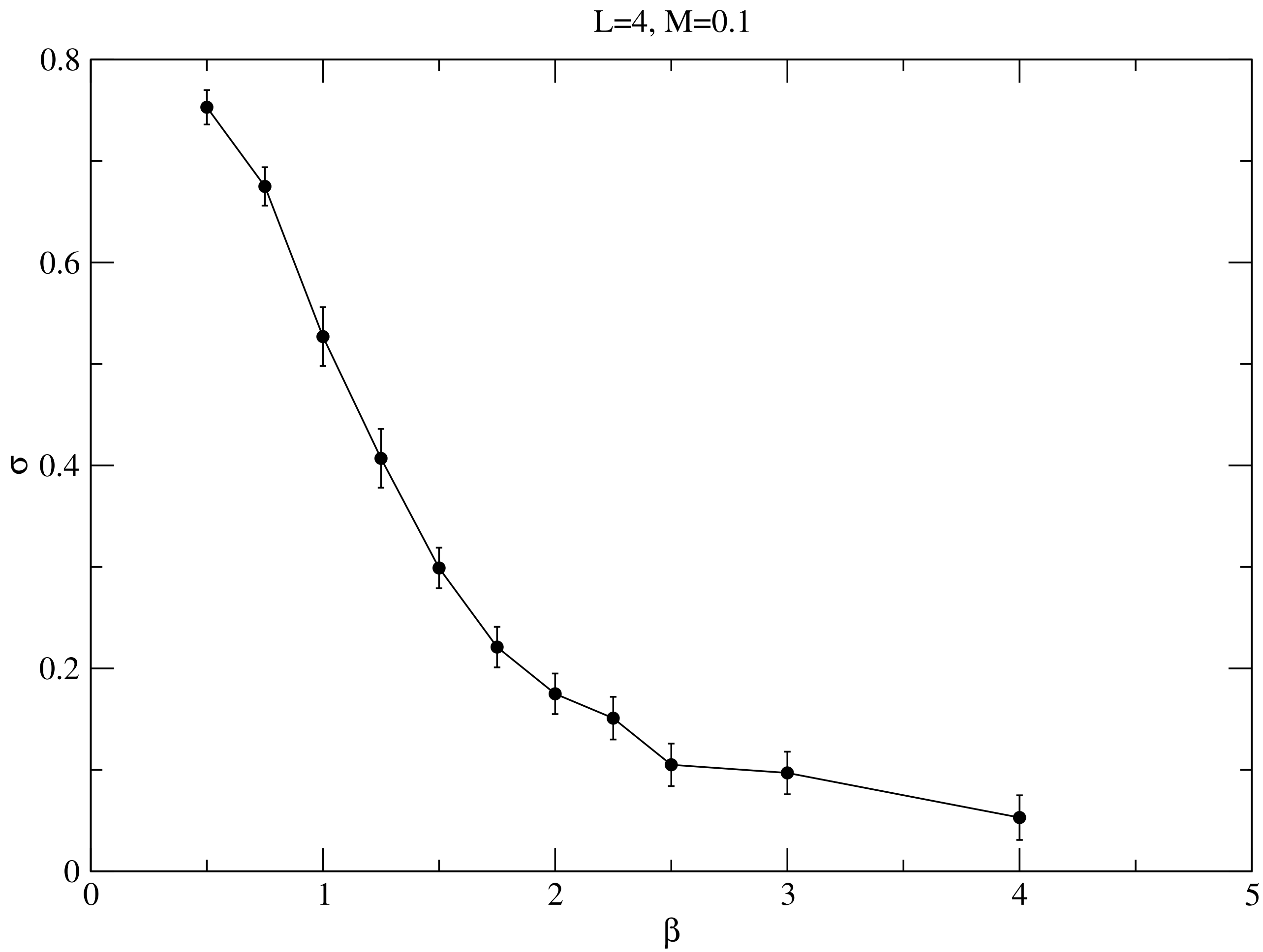
Q-symmetry allows us to compute bosonic action exactly

$$\beta < S_B >= \frac{3}{2} R L^2 \text{ independent of } \beta!$$

$(< S >=< Q\Lambda >= 0)$


m=0.1
L=2
L=4
L=8
1/(6L^2)<S_B>
beta
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5


L=4, M=0.1
50
40
30
20
10
0
S_G
0
1
2
3
4
5
β


L=4, M=0.1
0.8
0.6
0.4
0.2
0
σ
0
1
2
3
4
5
β

Twisting $\mathcal{N} = 4$ SYM in D=4

4 Majorana spinors Ψ^I_α. Kähler-Dirac twist:

$$SO(4)' = \mathrm{diag}(SO(4) \times SO(4)_R)$$

Regard supercharges and fermions as matrices:

$$\begin{aligned} \Psi &= \eta I + \psi_\mu \gamma_\mu + \frac{1}{2!}\chi_{\mu\nu}\gamma_\mu\gamma_\nu + \\ &+ \frac{1}{3!}\theta_{\mu\nu\lambda}\gamma_\mu\gamma_\nu\gamma_\lambda + \frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho \end{aligned}$$

16 real components needed for single Kähler-Dirac field!

Twisted fermions paired with superpartners $(\overline{\phi}, A_\mu, B_{\mu\nu}, W_{\mu\nu\lambda}, C_{\mu\nu\lambda\rho})$ through scalar Q supersymmetry (B and C multipliers)

4D Q transformations

Simple generalization of D=2

$$
\begin{aligned}
Q\overline{\phi} &= \eta \;\; Q\eta = [\phi, \overline{\phi}] \\
QA_\mu &= \psi_\mu \;\; Q\psi_\mu = -D_\mu \phi \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}] \;\; Q\chi_{\mu\nu} = B_{\mu\nu} \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} \;\; Q\theta_{\mu\nu\lambda} = [\phi, W_{\mu\nu\lambda}] \\
QC_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}] \;\; Q\kappa_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} \\
Q\phi &= 0
\end{aligned}
$$

Notice $Q^2 = \delta^\phi$

Gauge Fermion of $\mathcal{N} = 4$ SYM

$S = \beta Q \Lambda$ with

$$\begin{aligned}
\Lambda &= \int d^4x \mathrm{Tr}\left[\chi_{\mu\nu}\left(F_{\mu\nu} + \frac{1}{2}B_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]\right.\right. \\
&+ \left. D_\lambda W_{\lambda\mu\nu}\right) \\
&+ \psi_\mu D_\mu \overline{\phi} + \frac{1}{4}\eta[\phi, \overline{\phi}] + \frac{1}{3!}\theta_{\mu\nu\lambda}[W_{\mu\nu\lambda}, \overline{\phi}] \\
&+ \left.\frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\left(\sqrt{2}D_{[\mu}W_{\nu\lambda\rho]} + \frac{1}{2}C_{\mu\nu\lambda\rho}\right)\right]
\end{aligned}$$

Structure similar to $\mathcal{N} = 2$ in SYM $D = 2$

Carry out Q-variation and integrate out $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$

$$S = \beta\left(S_B + S_F + S_Y\right)$$

Continuum geometric action

$$
\begin{aligned}
S_F &= \int d^4x \mathrm{Tr}\left[-\chi_{\mu\nu} D_{[\mu}\psi_{\nu]} - \chi_{\mu\nu} D_\lambda \theta_{\lambda\mu\nu}\right. \\
&- \left.\eta D_\mu \psi_\mu - \frac{\sqrt{2}}{4!}\kappa_{\mu\nu\lambda\rho} D_{[\mu}\theta_{\nu\lambda\rho]}\right]
\end{aligned}
$$

$$
\begin{aligned}
S_B &= \int d^4x \mathrm{Tr}\left[-\frac{1}{2}\left(\left(F_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]\right)^2\right.\right. \\
&+ \left.(D_\lambda W_{\lambda\mu\nu})^2 + \frac{2}{4!}\left(D_{[\mu} W_{\nu\lambda\rho]}\right)^2\right) \\
&- \left.D_\mu \phi D_\mu \overline{\phi} + \frac{1}{4}[\phi, \overline{\phi}]^2 - \frac{1}{3!}[\phi, W_{\mu\nu\lambda}][\overline{\phi}, W_{\mu\nu\lambda}]\right]
\end{aligned}
$$

$$S_Y = \ldots$$

Connection to Marcus twisting

Is this action $\mathcal{N} = 4$ SYM ?

(not appear to be standard topological twist)

Change variables:

$$\begin{aligned} W_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho} V_\rho \\ \theta_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho} \overline{\psi}_\rho \\ \kappa_{\mu\nu\lambda\rho} &= \epsilon_{\mu\nu\lambda\rho} \overline{\eta} \end{aligned}$$

rescale fermions ...

Obtain a well-known topological twist of $\mathcal{N} = 4$ SYM due to Marcus.

Spinor formulation

Making another change of variables

$$\begin{aligned} X^\mu &= V_\mu \quad \mu = 0 \ldots 3 \\ X^4 &= \phi_1 \\ X^5 &= \phi_2 \end{aligned}$$

$$S_B = -\frac{1}{2} F_{\mu\nu}^2 - \left(D_\mu X^i\right)^2 - \frac{1}{2} \sum_{ij} [X_i, X_j]^2$$

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}$$

$$S_F = \frac{1}{2} \sum_{\alpha=1,2} \lambda_\alpha^\dagger \gamma . D \lambda_\alpha$$

where λ^1 and λ^2 columns of KD matrix.

Lattice $\mathcal{N} = 4$ in $D = 4$

Use same presciption as for $\mathcal{N} = 2$ SYM in $D = 2$.

- Lattice p-form fields on elementary p-cubes. Complexify.
- Gauge transformation props similar to $D = 2$
- Replace d and $d^{\dagger}$ by appropriate covariant lattice difference operators (generalize results of $D = 2$)
- Modify commutators by point splitting to maintain G.I

Again, to target $\mathcal{N} = 4$ SYM we will need to argue that Ward identities are satisfied as $\beta \to \infty$ after truncating the P.I to the real line.

Conclusions

- Theories with extended SUSY (in Eucl. space) can (often) be discretized while preserving G.I and a (twisted) SUSY
- Lattice theories are local and free of spectrum doubling.
- Discretization proceeds from reformulation in geometrical terms (Dirac-Kähler fields)
- Should be possible on curved spaces (simplicial manifolds)
- Truly non-perturbative formulation – numerical simulation possible

To do ...

- Pert. checks of W.I after projection to real line – both in D=2 and D=4 theories
- $\mathcal{N} = 4, 8$ theories in $D = 2$ and $\mathcal{N} = 8$ theory in $D = 3$ should be expressable in terms of KD twisted fields
- Simulations ... check Q-supersymmetry, phase of fermion det, study restoration of other supersymmetries, fine tuning etc
- Make contact with black hole physics ?