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Hybrid algorithm for an $\alpha-$nonexpansive mapping in a Banach space

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`This paper is dedicated to Professor Ljubomir Ćirić`

`Communicated by Professor V. Berinde`

Abstract

In this paper, we prove strong convergence theorem by the hybrid method for an $\alpha-$nonexpansive mapping in a Banach space. Our results complement and enrich the research contents of $\alpha-$nonexpansive mapping. Simultaneously, our main result generalizes Takahashi, Takeuchi, Kubota's result[W. Takahashi, Y. Takeuchi , R. Kubota, J. Math. Anal. Appl. 341 (2008) 276-286].

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1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\|, \quad for\ each\ x,\ y \in C. \tag{1.1}$$

Lots of iterative schemes for nonexpansive mappings have been introduced (see [1], [2], [3]), furthermore, many strong convergence theorems for nonexpansive mapping have been proved. On the other hand, there

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are many nonlinear mappings which are more general than the nonexpansive mapping. Compare to the exist problem of fixed point of those mapping, the iterative methods for finding fixed point is also very useful in studying in the fixed point theory and the theory of equations in other fields.

In 2007, Gobel and Pineda [4] introduced and studied a new mapping, called $\alpha-$nonexpansive mapping. The mapping is more general than the nonexpansive one.

Definition 1.1. For a given multiindex $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ satisfies $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n$ and $\sum_{i=1}^{n} \alpha_i = 1$. A mapping $T : C \to C$ is said to be $\alpha-$nonexpansive if

$$\sum_{i=1}^{n} \alpha_i \|T^i x - T^i y\| \leq \|x - y\|, \quad \forall\ x,\ y \in C. \tag{1.2}$$

In order to show that the class of $\alpha-$nonexpansive mapping is more general than the one of nonexpansive mappings, we give an example.

Example 1.2. Let $E = \mathbb{R}^1$, and

$$T(x) = \begin{cases} 0, & if \quad x = 0; \\ \frac{1}{x}, & if \quad x \in (0, +\infty). \end{cases}$$

Then, T is not nonexpansive but $\alpha-$nonexpansive.

Proof. Obviously, T is not nonexpansive. Taking $x = \frac{1}{2}$, $y = 0$, by the definition of Tx, we have

$$\|Tx - Ty\| = |2 - 0| > |\frac{1}{2} - 0| = \|x - y\|.$$

On the other hand, for every $x,\ y \in [0, +\infty)$, we have

$$\|T^2 x - T^2 y\| = \|x - y\|.$$

Therefore, we can affirm that

$$0\|Tx - Ty\| + \|T^2 x - T^2 y\| = \|x - y\|$$

where $\alpha = (\alpha_1, \alpha_2) = (0, 1)$. Then T is an $\alpha-$nonexpansive mapping but not a nonexpansive one. □

If T is nonexpansive self-mapping, we can imply that T must be $\alpha-$nonexpansive one, where $\alpha = (\alpha_1,\ \alpha_2, \cdots, \alpha_n) = (\frac{1}{n},\ \frac{1}{n}, \cdots, \frac{1}{n},)$.

For technical reason we always assume that the first coefficient α_1 is nonzero, that is, $\alpha_1 > 0$. In this case the mapping T satisfies the Lipschitz condition

$$\|Tx - Ty\| \leq \frac{1}{\alpha_1}\|x - y\|, \quad \forall\ x,\ y \in C. \tag{1.3}$$

Recall a mapping $T : C \to C$ is said to be an asymptotically nonexpansive mapping, if exists a sequence of reals $\{\gamma_n\}$ with $0 \leq \gamma_n \to 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \gamma_n)\|x - y\|, \quad \forall\ x,\ y \in C. \tag{1.4}$$

Noticing that, with regard to (1.4), take $n = 1,\ 2,\ \cdots,\ n$, we have

$$\|Tx - Ty\| \leq (1 + \gamma_1)\|x - y\|,$$

$$\|T^2 x - T^2 y\| \leq (1 + \gamma_2)\|x - y\|,$$

$$\cdots$$

$\|T^n x - T^n y\| \leq (1+\gamma_n)\|x-y\|.$

By multiplying the above inequalities with $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n$, respectively, we have

$$\alpha_1\|Tx - Ty\| \leq \alpha_1(1+\gamma_1)\|x-y\| = \alpha_1\|x-y\| + \alpha_1\gamma_1\|x-y\|,$$

$$\alpha_2\|T^2x - T^2y\| \leq \alpha_2(1+\gamma_2)\|x-y\| = \alpha_2\|x-y\| + \alpha_2\gamma_2\|x-y\|,$$

$$\cdots$$

$$\alpha_n\|T^nx - T^ny\| \leq \alpha_n(1+\gamma_n)\|x-y\| = \alpha_n\|x-y\| + \alpha_n\gamma_n\|x-y\|.$$

Adding all the inequalities above, we obtain

$$\sum_{i=1}^{n} \alpha_i\|T^ix - T^iy\| \leq \sum_{i=1}^{n} \alpha_i\|x-y\| + \sum_{i=1}^{n} \alpha_i\gamma_i\|x-y\|.$$

Since $\sum_{i=1}^{n} \alpha_i = 1$ and suppose that $\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i\gamma_i = 0$, we have

$$\sum_{i=1}^{n} \alpha_i\|T^ix - T^iy\| \leq (1+\lambda_n)\|x-y\|, \tag{1.5}$$

where $\lambda_n = \sum_{i=1}^{n} \alpha_i\gamma_i$.

Definition 1.3. For a given multiindex $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ satisfies $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n$ and $\sum_{i=1}^{n} \alpha_i = 1$. A mapping $T : C \to C$ is said to be $\alpha-$mean-asymptotically-nonexpansive if exists a sequence of reals $\{\lambda_n\}$ with $0 \leq \lambda_n \to 0$ such that

$$\sum_{i=1}^{n} \alpha_i\|T^ix - T^iy\| \leq (1+\lambda_n)\|x-y\|.$$

Remark 1.4. From the analysis above and Definition 1.1 , we know that each $\alpha-$nonexpansive mapping is an $\alpha-$mean-asymptotically-nonexpansive one.

The last observation is that (1.2) implies that for $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ the mapping

$$T_\alpha x = \sum_{i=1}^{n} \alpha_i T^i x, \quad \forall\ x \in C \tag{1.6}$$

is nonexpansive. However, nonexpansiveness of T_α is much weaker than (1.2), for instance, it doesn't entail the continuity of T (see [4]).

Recently, Chakkrid Klin-eam, Suthep Suantai[5], introduced the relation of the fixed point sets between $\alpha-$nonexpasive operator and T_α operator. they give the following theorem:

Theorem 1.5. *(see Theorem 3.1 of Klin-eam, Suantai [5]) Let C be a closed convex subset of a Banach space E and for all $n \in \mathbb{N}$, let $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ such that $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n,\ \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an $\alpha-$nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$, then $F(T) = F(T_\alpha)$, where $F(T)$ is the fixed point set of T.*

At the same time, they have succeeded in proving the demiclosedness principle for the $\alpha-$nonexpansive mappings.

Theorem 1.6. *(see Theorem 3.4 of Klin-eam, Suantai [5]) Let C be a closed convex subset of a Banach space E and for all $n \in \mathbb{N}$, let $\alpha = (\alpha_1, \ \alpha_2, \ \cdots, \ \alpha_n)$ such that $\alpha_i \geq 0, \ i = 1, \ 2, \ \cdots, \ n, \ \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an $\alpha-$nonexpansive mapping from C into itself with $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$, if $\{x_n\} \subset C$ converges weakly to x and $\{x_m - Tx_m\}$ converges strongly to 0 as $n \to \infty$, then $x \in F(T)$.*

Motivated by results above, we prove strong convergence theorems by the hybrid method for an $\alpha-$nonexpansive mapping in a Banach space.

2. Preliminaries

In what follows, E denotes a real Banach space with norm $\|\cdot\|$ and E^* the dual space of E. The norm of E^* is also denoted by$\|\cdot\|$. For $x^* \in E^*$, its value at $x \in E$ is denoted by $\langle x, x^* \rangle$. The normalized duality mapping J from E to E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.1}$$

for all $x \in E$. Some properties of the duality mapping have been given in [6, 7].

A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for all $x, \ y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. We also know that if E is a uniformly convex Banach space, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E, then the Banach space E is said to be smooth if

$$\lim_{n\to\infty} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, \ y \in U$.

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Then, for arbitrarily fixed $x \in E$, a function $C \ni \|y - x\| \in \mathbb{R}$ has a unique minimizer $y_x \in C$. Using such a point, we define the metric projection P_C by $P_C x = y_x = argmin_{y\in C}\|x - y\|^2$ for every $x \in E$. In a similar fashion, we can see that a function $C \ni y \mapsto \phi(x, y) \in \mathbb{R}$ also has a unique minimizer $z_x \in C$. The generalized projection Π_C of E onto C is defined by $\Pi_C x = z_x = argmin_{y\in C}\phi(x, y)$ for every $x \in E$; see [8].

In fact, we have the following result.

Lemma 2.1. *(see [8]) Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = min\{\phi(z, x) : z \in C\}$.*

In order to prove our results, the following lemmas are needed.

Lemma 2.2. *(see [8]) Let C be a nonempty closed and convex subset of a real smooth Banach space E, Let $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx_0 - Jx \rangle \geq 0, \quad \forall z \in C.$$

In Chakkrid Klin-eam, Suthep Suantai [5], they also give the following lemma:

Lemma 2.3. *(see Lemma 3.3 of Klin-eam, Suantai [5]) Let C be a closed convex subset of a Banach space E and for all $n \in \mathbb{N}$, let $\alpha = (\alpha_1, \ \alpha_2, \ \cdots, \ \alpha_n)$ such that $\alpha_i \geq 0, \ i = 1, \ 2, \ \cdots, \ n, \ \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an $\alpha-$nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$. Let $\{x_m\}$ be a bounded sequence in C, then $\|x_m - Tx_m\| \to 0$ if and only if $\|x_m - T_\alpha x_m\| \to 0$ as $m \to \infty$.*

A mapping $T : C \to C$ is said to be closed, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Lemma 2.4. *Let C be a closed convex subset of a Banach space E and for all $n \in \mathbb{N}$, let $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ such that $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n,\ \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an $\alpha-$nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$. Let $\{x_m\} \subset C$ converges strongly to x and $\{x_n - Tx_m\}$ converges strongly to 0 as $n \to \infty$, then $x \in F(T)$.*

Proof. Since $\{x_n - Tx_m\}$ converges strongly to 0 as $n \to \infty$, from Lemma 2.3, we have $\{x_m - T_\alpha x_m\}$ converges stronglly to 0 as $n \to \infty$. Since T_α is a nonexpansive mapping, which is closed. We have $x \in F(T_\alpha)$. From Theorem 1.5, we obtain that $x \in F(T)$. □

Lemma 2.5. *Let C be a uniformly convex and smooth Banach space E and for all $n \in \mathbb{N}$, let $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ such that $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n,\ \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an $\alpha-$nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$. Then $F(T)$ is closed and convex.*

Proof. By Theorem 1.5, we have $F(T) = F(T_\alpha)$. Since T_α is a nonexpansive mapping, we know that $F(T_\alpha)$ is closed and convex here. So, with regard to an $\alpha-$nonexpansive T, we also have that $F(T)$ is closed and convex. □

3. The Main Result

In this section, we consider a strong convergence theorem for an $\alpha-$nonexpansive operator in a Banach space.

Theorem 3.1. *Let C be a closed convex subset of a uniformly convex and smooth Banach space E. Let $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ such that $\alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ n,\ \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, let $T : C \to C$ be an $\alpha-$nonexpansive mapping such that $F(T) \neq \emptyset$. For $x_0 \in E$, $C_1 = C$ and $x_1 = \Pi_{C_1} x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $0 < a \leq \beta_n \leq 1$, for all $n \in \mathbb{N}$. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$, Then, x_n converges strongly to $z_0 = \Pi_{F(T)} x_0$.

Proof. Firstly, we show that C_n is closed and convex and $F(T) \subset C_n$ for every $n \in \mathbb{N}$

It is obvious that C_n is closed, for every $n \in \mathbb{N}$. It follows that C_n is convex for every $n \in \mathbb{N}$. Since

$$\|y_n - z\| \leq \|x_n - z\| \tag{3.2}$$

is equivalent to

$$\|y_n - x_n\|^2 + 2\langle y_n - x_n, Jx_n - Jz\rangle \leq 0. \tag{3.3}$$

So, C_n is convex for every $n \in \mathbb{N}$. For $u \in F(T)$, we compute that

$$\begin{aligned} \|y_n - u\| &= \|(1-\beta_n)x_n + \beta_n T x_n - u\| \\ &\leq (1-\beta_n)\|x_n - u\| + \beta_n \|T x_n - u\| \\ &= (1-\beta_n)\|x_n - u\| + \beta_n \|T x_n - T_\alpha u\| \\ &= (1-\beta_n)\|x_n - u\| + \beta_n \|\alpha_1 (T x_n - T u) \\ &\quad + \alpha_2 (T x_n - T^2 u) + \ \cdots \ + \alpha_n (T x_n - T^n u)\| \\ &\leq (1-\beta_n)\|x_n - u\| + \beta_n \frac{1-\alpha_1^{n-1}}{\alpha_1^{n-1}} \|x_n - Tu\| \\ &\leq (1-\beta_n)\|x_n - u\| + \beta_n \|x_n - u\| \\ &= \|x_n - u\|. \end{aligned} \tag{3.4}$$

It implies that $u \in C_n$, for each $n \in \mathbb{N}$. So, we have $F(T) \subset C_n$ for all $n \in \mathbb{N}$.

Next, we prove that $\{x\}$ is bounded. Since $F(T)$ is a nonempty closed convex subset of C, there exists a unique element $z_0 \in F(T)$ such that $z_0 = \Pi_{F(T)}x_0$. From $x_{n+1} = \Pi_{C_{n+1}}x_0$, we have

$$\|x_{n+1} - x_0\| \le \|z - x_0\|, \tag{3.5}$$

for every $z \in C_{n+1}$. As $z_0 \in F(T) \subset C_{n+1}$, we obtain

$$\|x_{n+1} - x_0\| \le \|z_0 - x_0\|, \tag{3.6}$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded.

On the other hand, $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$, we have

$$x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n,$$

for all $n \in \mathbb{N}$. Since $x_n = \Pi_{C_n}x_0$, we obtain that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \tag{3.7}$$

for all $n \in \mathbb{N}$. It follows from (3.6) that the limit of $\{x_n - x_0\}$ exists.

Since $C_m \subset C_n$, $x_{m+1} = \Pi_{C_{m+1}}x_0 \in C_m \subset C_n$ for all $m \ge n$ and $x_{m+1} = \Pi_{C_{m+1}}x_0$, in view of Lemma 2.1, one has

$$\langle x_{n+1} - x_0, Jx_{n+1} - Jx_{n+1}\rangle \ge 0. \tag{3.8}$$

We compute that

$$\begin{aligned}
\|x_{m+1} - x_{n+1}\|^2 &= \|x_{m+1} - x_0 - (x_{n+1} - x_0)\|^2 \\
&= \|x_{m+1} - x_0\|^2 + \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, Jx_{m+1} - Jx_0\rangle \\
&= \|x_{m+1} - x_0\|^2 + \|x_{n+1} - x_0\|^2 \\
&\quad - 2\langle x_{n+1} - x_0, Jx_{m+1} - Jx_{n+1} + Jx_{n+1} - Jx_0\rangle \\
&= \|x_{m+1} - x_0\|^2 - \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, Jx_{m+1} - Jx_{n+1}\rangle \\
&\le \|x_{m+1} - x_0\|^2 - \|x_{n+1} - x_0\|^2.
\end{aligned}$$

Since the limit of $\|x_{n+1} - x_0\|$ exists, we get

$$\lim_{n\to\infty} \|x_m - x_n\| = 0. \tag{3.9}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since Banach space is a complete metric space and C is closed and convex, one can assume that

$$x_n \to p, \quad as\ n \to \infty. \tag{3.10}$$

Noticing that $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we obtain that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|. \tag{3.11}$$

In view of (3.10), we have that

$$\|y_n - x_{n+1}\| \to 0, \quad as\ n \to \infty \tag{3.12}$$

and

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad as\ n \to \infty. \tag{3.13}$$

From $y_n = (1 - \beta_n)x_n + \beta_n Tx_n$, we have

$$\|x_n - Tx_n\| = \frac{1}{\beta_n}\|y_n - x_n\|. \tag{3.14}$$

Because of assumption that $0 < a \leq \beta_n \leq 1$, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.15}$$

This implies from Lemma 2.4 that $p \in F(T)$.

Finally, we prove that $p = z_0 = \Pi_{F(T)}x_0$.

From $x_{n+1} = \Pi_{C_{n+1}}x_0$ and Lemma 2.1, we have

$$\langle Jx_0 - Jx_{n+1}, x_{n+1} - y\rangle \geq 0, \tag{3.16}$$

for all $y \in C_{n+1}$. Since $F(T) \subset C_{n+1}$, we also have that

$$\langle Jx_0 - Jx_{n+1}, x_{n+1} - q\rangle \geq 0, \tag{3.17}$$

for all $q \in F(T)$. By taking limit in (3.17), one has

$$\langle Jx_0 - Jx_{n+1}, p - q\rangle \geq 0, \tag{3.18}$$

for all $q \in F(T)$. Now, by Lemma 2.1 again, we have that

$$p = z_0 = \Pi_{F(T)}x_0.$$

The proof is completed. □

Because every nonexpansive mapping is $\alpha-$nonexpansive, the following corollary can be obtain by Theorem 3.1.

Corollary 3.2. *Let C be a closed convex subset of a uniformly convex and smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For $x_0 \in E$, $C_1 = C$ and $x_1 = \Pi_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \beta_n \leq 1$, for all $n \in \mathbb{N}$. Then, x_n converges strongly to $z_0 = \Pi_{F(T)}x_0$.

Remark 3.3. Corollary 3.2 is equivalence to Theorem 4.1 of Takahashi, Takeuchi, Kubota [9]. Therefore, Our main result Theorem 3.1 is more general than Theorem 4.1 [9].

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