A CLASS OF ANALYTIC FUNCTIONS BASED ON CONVOLUTION

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Abstract. We introduce a class $T_{S}^{p}(\alpha)$ of analytic functions with negative coefficients defined by convolution with a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} b_{n} z^{n}$, $b_{n} > 0$, $|z| < 1$. We obtain the coefficient inequality, coefficient estimate, distortion theorem, a convolution result, extreme points and integral representation for functions in the class $T_{S}^{p}(\alpha)$.

1 Introduction

Let $S$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n} \quad (1.1)$$

that are analytic and univalent in the unit disk $\Delta = \{ z : |z| < 1 \}$. Let $S^{*}(\alpha)$ denote the subfamily of $S$ consisting of functions starlike of order $\alpha$, $0 \leq \alpha < 1$.

Let $T$ be the class of all analytic functions with negative coefficients of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_{n} z^{n} \quad (a_{n} \geq 0, z \in \Delta) \quad (1.2)$$

The subclass of $T$ consisting of starlike functions of order $\alpha$ denoted by $T^{*}(\alpha) = T \cap S^{*}(\alpha)$ was studied by Silverman [7]. In [8], the subclass $T_{S}^{p}(\alpha)$ of functions of the form (1.2) for which

$${\text{Re}} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \geq \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad -1 \leq \alpha < 1$$

has been studied. We need the following theorem proved in [8].

Theorem 1. A necessary and sufficient condition for $f$ of the form $f(z) = z - \sum_{n=2}^{\infty} a_{n} z^{n}$, $a_{n} \geq 0$ to be in $T_{S}^{p}(\alpha)$, $-1 \leq \alpha < 1$ is that $\sum_{n=2}^{\infty} [2n - (\alpha + 1)] a_{n} \leq 1 - \alpha$.

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In [1], Ahuja has studied the class $T_\lambda(\alpha)$, consisting of functions $f(z) \in T$ satisfying the condition $\Re \left( \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} \right) > \alpha$, $z \in \Delta, \lambda > -1, \alpha < 1$, where the operator $D^\lambda f$ is the Ruscheweyh derivative [5] of $f$ defined by $D^\lambda f(z) = f(z) \ast \frac{z}{(1-z)^{\lambda+1}}$. In fact we can write

$$T_\lambda(\alpha) = \left\{ f \in T : \Re \left( \frac{z(f \ast g)'(z)}{(f \ast g)(z)} \right) > \alpha, \ g(z) = \frac{z}{(1-z)^{\lambda+1}} \right\},$$

where $f \ast g$ is the Hadamard product of analytic functions.

Recall that for any two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f \ast g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g \ast f)(z).$$

Classes of analytic functions defined by convolution with given analytic functions have been investigated for their properties (see for example [2, 3, 4, 9] to mention a few). In this paper, motivated by [1], we introduce and study a class $TS^0_p(\alpha)$, $-1 \leq \alpha < 1$ where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n > 0, z \in \Delta$ is a fixed analytic function.

**Definition 2.** Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be a fixed analytic function in $\Delta = \{ z : |z| < 1 \}$ and for $n \geq 2, b_n > 0$. The class $TS^0_p(\alpha)$ consists of analytic functions of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where $z \in \Delta$ and for $n \geq 2, a_n \geq 0$, satisfying the inequality,

$$\Re \left\{ \frac{z(f \ast g)'(z)}{(f \ast g)(z)} - \alpha \right\} \geq \left| \frac{z(f \ast g)'(z)}{(f \ast g)(z)} - 1 \right|, \quad -1 \leq \alpha < 1$$

We obtain coefficient inequality, coefficient estimate, distortion theorem, convolution result, extreme points and integral representation for functions in the class $TS^0_p(\alpha)$. Unless mentioned otherwise, the function $g$ is taken as in definition 2.

## 2 The class $TS^0_p(\alpha)$

We begin with a necessary and sufficient condition to be in the class $TS^0_p(\alpha)$.

**Theorem 3.** A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in TS^0_p(\alpha)$ if and only if $\sum_{n=2}^{\infty} |2n - (\alpha + 1)| a_n b_n \leq 1 - \alpha$. 

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Proof. From the definitions of the classes $T S^g_\alpha$ and $T S^p_\alpha$ it follows on using Theorem 1 that

$$f \in T S^g_\alpha \Leftrightarrow f * g \in T S^p_\alpha \Leftrightarrow \sum_{n=2}^{\infty} [2n - (\alpha + 1)] a_n b_n \leq 1 - \alpha.$$ 

\[\square\]

Remark 4. For $-1 \leq \alpha < 1$ and $g(z) = \frac{z}{1-z}$, we have $T S^g_\alpha = T S^p_\alpha$ [8]

Theorem 5. If $f \in T S^g_\alpha$, then

$$a_n \leq \frac{1 - \alpha}{[2n - (\alpha + 1)] b_n}$$

with equality only for the functions of the form

$$f_n(z) = z - \frac{1 - \alpha}{[2n - (\alpha + 1)] b_n} z^n.$$

Proof. If $f \in T S^g_\alpha$, then

$$[2n - (\alpha + 1)] a_n b_n \leq \sum_{n=2}^{\infty} [2n - (\alpha + 1)] a_n b_n \leq 1 - \alpha$$

implies $a_n \leq \frac{1 - \alpha}{[2n - (\alpha + 1)] b_n}$.

We have equality for

$$f_n(z) = z - \frac{1 - \alpha}{[2n - (\alpha + 1)] b_n} z^n \in T S^g_\alpha.$$

\[\square\]

Theorem 6. If $f \in T S^g_\alpha$, then

$$r - \frac{1 - \alpha}{\min[2n-1-\alpha] b_n} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{\min[2n-1-\alpha] b_n} r^2,$$

$|z| = r < 1$. The result is sharp for $f(z) = z - \frac{1 - \alpha}{\min[2n-1-\alpha] b_n} z^2$.
Proof. Let \( |z| = r < 1 \). For \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), we have

\[
|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^{2n} = r + r^2 \sum_{n=2}^{\infty} a_n
\]

(2.1)

Since \( \min[(2n - 1 - \alpha)b_n] \leq [2n - (\alpha + 1)]b_n \), in view of Theorem 3,

\[
\min[(2n - 1 - \alpha)b_n] \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} [2n - (\alpha + 1)]a_n b_n \leq 1 - \alpha
\]

Hence

\[
|f(z)| \leq r + r^2 \frac{1 - \alpha}{\min[(2n - 1 - \alpha)b_n]}.
\]

Similarly we can show that \( |f(z)| \geq r - r^2 \frac{1 - \alpha}{\min[(2n - 1 - \alpha)b_n]} \).

We now prove a convolution result for the family \( T S^p_\alpha(\alpha) \).

Definition 7. Let

\[
f_1(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0
\]

and

\[
f_2(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n > 0.
\]

Then the Quasi Hadamard product \((f_1 * f_2)(z)\) is defined by

\[
(f_1 * f_2)(z) = f_1(z) * f_2(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n
\]
Theorem 8. Let $f \in TS^{k_1}_p(\alpha)$, $g \in TS^{k_2}_p(\alpha)$ where, for $z \in \Delta$,
\begin{align*}
k_i(z) &= z + \sum_{n=2}^{\infty} b_{in} z^n, \quad b_{in} > 0 \text{ for } i = 1, 2 \\
h(z) &= z + \sum_{n=2}^{\infty} h_n z^n, \quad h_n > 0 \\
f(z) &= z - \sum_{n=2}^{\infty} f_n z^n, \quad f_n > 0 \\
g(z) &= z - \sum_{n=2}^{\infty} g_n z^n, \quad g_n > 0
\end{align*}
then
$$f * g \in TS^{h}_p(\gamma), \quad \gamma = \min_n G(n)$$
where
$$G(n) = \frac{[2n - (\alpha + 1)] [2n - (\beta + 1)] b_{1n} b_{2n} - (2n - 1) h_n (1 - \alpha)(1 - \beta)}{[2n - (\alpha + 1)] [2n - (\beta + 1)] b_{1n} b_{2n} - h_n (1 - \alpha)(1 - \beta)}$$
provided $b_{1n} b_{2n} > (2n - 1) h_n$.

Proof. Since $f \in TS^{k_1}_p(\alpha)$, we have
$$\sum_{n=2}^{\infty} \frac{[2n - (\alpha + 1)]}{1 - \alpha} f_{n1} b_{1n} \leq 1 \quad (2.2)$$
Also $g \in TS^{k_2}_p(\beta)$ implies
$$\sum_{n=2}^{\infty} \frac{[2n - (\beta + 1)]}{1 - \beta} g_{n2} b_{2n} \leq 1 \quad (2.3)$$
Now, we will show that
$$f * g = z - \sum_{n=2}^{\infty} f_n g_n z^n \in TS^{h}_p(\gamma) \text{ i.e. } \sum_{n=2}^{\infty} \frac{[2n - (\gamma + 1)]}{1 - \gamma} f_{n1} g_{n} h_{n} \leq 1. \quad (2.4)$$
In order to prove that (2.2) and (2.3) imply (2.4), we note that
$$\sum_{n=2}^{\infty} \left[ \frac{[2n - (\alpha + 1)] [2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} f_{n1} b_{1n} b_{2n} \right]^{1/2} \leq \left( \sum_{n=2}^{\infty} \frac{[2n - (\alpha + 1)]}{1 - \alpha} f_{n1} b_{1n} \right)^{1/2} \left( \sum_{n=2}^{\infty} \frac{[2n - (\beta + 1)]}{1 - \beta} g_{n2} b_{2n} \right)^{1/2} \leq 1 \quad (2.5)$$
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In order to prove (2.4), it suffices to show that
\[
\frac{2n - (\gamma + 1)}{1 - \gamma} f_n g_n h_n \leq \left[ \frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} f_n g_n b_{1n} b_{2n} \right]^{1/2}
\]
or equivalently
\[
\sqrt{f_n g_n} \leq \frac{1 - \gamma}{2n - (\gamma + 1)} h_n \left[ \frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} b_{1n} b_{2n} \right]^{1/2}
\]
From (2.5), we have
\[
\sqrt{f_n g_n} \leq \sqrt{\frac{1 - \alpha)(1 - \beta)}{2n - (\alpha + 1)[2n - (\beta + 1)] b_{1n} b_{2n}}
\]
(2.6)
In view of (2.6) and (2.7), it is enough to prove that
\[
\sqrt{\frac{1 - \alpha)(1 - \beta)}{2n - (\alpha + 1)[2n - (\beta + 1)] b_{1n} b_{2n}} \leq \frac{2n - (\gamma + 1)}{1 - \gamma} \left[ \frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} b_{1n} b_{2n} \right]^{1/2}
\]
\[
\Leftrightarrow \frac{2n - (\gamma + 1)}{1 - \gamma} \leq \frac{1}{h_n} \sqrt{\frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} b_{1n} b_{2n}} \sqrt{\frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} b_{1n} b_{2n}}
\]
\[
\Leftrightarrow \frac{2n - (\gamma + 1)}{1 - \gamma} \leq \frac{1}{h_n} \left[ \frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} b_{1n} b_{2n} \right]^{1/2}
\]
\[
\Leftrightarrow 2n - (\gamma + 1) \leq B - B\gamma
\]
where
\[
B = \frac{1}{h_n} \frac{2n - (\alpha + 1)[2n - (\beta + 1)]}{(1 - \alpha)(1 - \beta)} b_{1n} b_{2n}.
\]
Then \(B \geq 2n - 1 > 1\) and (2.8) is equivalent to
\[
\gamma \leq \frac{B - (2n - 1)}{B - 1}
\]
\[
= \frac{1 - \frac{2n - 1}{B}}{1 - \frac{1}{B}}
\]
\[
= \frac{1 - \frac{2n - (\alpha + 1)[2n - (\beta + 1)] h_n b_{1n} b_{2n}}{h_n[1 - \alpha)(1 - \beta)} [2n - (\alpha + 1)[2n - (\beta + 1)] b_{1n} b_{2n}]}{1 - \frac{2n - (\alpha + 1)[2n - (\beta + 1)] h_n b_{1n} b_{2n}}{h_n[1 - \alpha)(1 - \beta)}} = G(n),
\]
(2.11)
This proves the result. \(\square\)
Analytic functions based on convolution

Theorem 9. If

\[ f_1(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad f_2(z) = z - \sum_{n=2}^{\infty} c_n z^n \]

and

\[ f_3(z) = z - \sum_{n=2}^{\infty} (a_n^2 + c_n^2) z^n \]

where \( f_1(z), f_2(z) \in TS^p_\alpha \), then \( f_3 \in TS^p_\beta \) where

\[ \beta = \min_n \left[ \frac{(2n - (\alpha + 1))^2}{2n - (\alpha + 1)} - 2(2n - 1)(1 - \alpha)^2 \right]^{\frac{1}{2}} \]

and \( g_1(z) = (g + g)(z) = z + \sum_{n=2}^{\infty} b_n^2 z^n \).

Proof. Since \( f_1(z) \in TS^p_\alpha \), we have

\[ \sum_{n=2}^{\infty} \frac{2n - (\alpha + 1)}{1 - \alpha} a_n b_n \leq 1 \]

and hence

\[ \sum_{n=2}^{\infty} \left( \frac{2n - (\alpha + 1)}{1 - \alpha} \right)^2 a_n^2 b_n^2 \leq 1. \]

Also since \( f_2(z) \in TS^p_\alpha \), we have

\[ \sum_{n=2}^{\infty} \frac{2n - (\alpha + 1)}{1 - \alpha} c_n b_n \leq 1 \]

and hence

\[ \sum_{n=2}^{\infty} \left( \frac{2n - (\alpha + 1)}{1 - \alpha} \right)^2 c_n^2 b_n^2 \leq 1. \]

Therefore

\[ \frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{2n - (\alpha + 1)}{1 - \alpha} \right)^2 (a_n^2 + c_n^2) b_n^2 \leq 1. \]

In view of Theorem 3, we have to show that

\[ \sum_{n=2}^{\infty} \left( \frac{2n - (\beta + 1)}{1 - \beta} \right) (a_n^2 + c_n^2) b_n^2 \leq 1 \]

The last inequality will be satisfied if

\[ \frac{2n - (\beta + 1)}{1 - \beta} \leq \frac{1}{2} \left( \frac{2n - (\alpha + 1)}{1 - \alpha} \right)^2. \]

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Solving for $\beta$, we have

$$\beta \leq \frac{[2n - (\alpha + 1)]^2 - 2(2n - 1)(1 - \alpha)^2}{[2n - (\alpha + 1)]^2 - 2(1 - \alpha)^2}.$$ 

Hence the result follows.

3 Extreme points and Integral representation

Theorem 10. Let

$$f_1(z) = z, \ f_n(z) = z - \frac{1 - \alpha}{[2n - (\alpha + 1)]b_n} z^n,$$

$n = 2, 3, \ldots$ then $f \in TS^g_p(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z);$$

where $\mu_n \geq 0$ and $\sum_{n=1}^{\infty} \mu_n = 1$. In particular the extreme points of $TS^g_p(\alpha)$ are the functions $f_1(z) = z$ and

$$f_n(z) = z - \frac{1 - \alpha}{[2n - (\alpha + 1)]b_n} z^n,$$

$n = 2, 3, \ldots$.

Proof. First let $f$ be expressed as in the above theorem. This means that we can write

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{(1 - \alpha)\mu_n}{[2n - (\alpha + 1)]b_n} z^n \Rightarrow z - \sum_{n=2}^{\infty} t_n z^n$$

Therefore $f \in TS^g_p(\alpha)$, since $\sum_{n=2}^{\infty} \frac{2n - (\alpha + 1)}{1 - \alpha} t_n b_n = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 < 1$

Conversely, If $f \in TS^g_p(\alpha)$, by Theorem 3, we have

$$a_n \leq \frac{1 - \alpha}{[2n - (\alpha + 1)]b_n}$$

so we may set $\mu_n = \frac{[2n - (\alpha + 1)]a_n b_n}{1 - \alpha}$ and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$.
Then
\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{2n - (\alpha + 1)} b_n z^n
\]
\[
= z - \sum_{n=2}^{\infty} \mu_n [z - f_n(z)]
\]
\[
= \left( 1 - \sum_{n=2}^{\infty} \mu_n \right) z + \sum_{n=2}^{\infty} \mu_n f_n(z) = \sum_{n=1}^{\infty} \mu_n f_n(z).
\]

\[\square\]

Remark 11.

1. For \(g(z) = \frac{z}{1 - z}\), we obtain corollary 1 in [8].

2. For \(g(z) = z\) and replacing \(\alpha\) by \(\frac{1 + \alpha}{2}\), in \(T^* (\alpha)\), we obtain the corresponding result in [7].

Theorem 12. If \(f\) is in \(TS_p^\alpha(\alpha)\), then
\[
(f \ast g)(z) = \exp \int_0^z \frac{1 + \alpha \rho(t)}{t(1 - \rho(t))} dt
\]
for some \(\rho(z)\), \(|\rho(z)| < 1\), \(z \in \Delta\) where
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n,
\]
\(b_n > 0\) for \(n = 2, 3, \ldots\). Also
\[
(f \ast g)(z) = z \exp \left[ \int_x \ln(1 - xz)^{-1 + \alpha} d\mu(x) \right]
\]
where \(\mu(x)\) is probability measure on \(X = \{x : |x| = 1\}\).

Proof. Let \(f \in TS_p^\alpha(\alpha)\) and
\[
\omega = \frac{z(f \ast g)'(z)}{(f \ast g)(z)}.
\]
We have \(\text{Re } \omega - \alpha > |\omega - 1|\). Therefore
\[
\left| \frac{\omega - 1}{\omega - \alpha} \right| < 1
\]
and
\[
\frac{\omega - 1}{\omega - \alpha} = \rho(z),
\]
where \(|\rho(z)| < 1, z \in \Delta\).

This gives
\[
\frac{(f * g)'(z)}{(f * g)(z)} = \frac{1 - \alpha \rho(z)}{z(1 - \rho(z))}
\]
and therefore
\[
(f * g)(z) = \exp\left[\int_0^z \frac{1 - \alpha \rho(t)}{t(1 - \rho(t))} \, dt\right].
\]

Now set \(X = \{x : |x| = 1\}\). Then we have \(\frac{\omega - 1}{\omega - \alpha} = xz\) and hence
\[
\frac{(f * g)'(z)}{(f * g)(z)} = \frac{1 - \alpha xz}{z(1 - xz)} = \frac{1}{z} + \frac{(1 - \alpha)x}{1 - xz}.
\]

Integrating we obtain,
\[
\frac{ln(f * g)(z)}{z} = (-1 + \alpha)ln(1 - xz).
\]

This proves the second representation
\[
(f * g)(z) = z \exp\left[\int_x ln(1 - xz)^{-1+\alpha} \, d\mu(x)\right].
\]

\(\square\)

**Remark 13.** Taking \(g(z) = \frac{z}{(1-z)^{1+\lambda}}, \lambda > -1\), \(TS_p^g(\alpha)\) reduces to the class \(D(\alpha, \beta, \lambda)\) of [6], so that Theorems 2.1, 2.4, 2.7 in [6] are consequences of our Theorems 3, 10, 12.

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**References**


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