FRACTIONAL ORDER DIFFERENTIAL INCLUSIONS ON THE HALF-LINE

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Abstract. We are concerned with the existence of bounded solutions of a boundary value problem on an unbounded domain for fractional order differential inclusions involving the Caputo fractional derivative. Our results are based on the fixed point theorem of Bohnnenblust-Karlin combined with the diagonalization method.

1 Introduction

This paper deals with the existence of bounded solutions for boundary value problems (BVP for short) for fractional order differential inclusions of the form

\[
c^D_\alpha y(t) \in F(t, y(t)), \quad t \in J := [0, \infty),
\]

\[
y(0) = y_0, \quad y \text{ is bounded on } J,
\]

where \(c^D_\alpha\) is the Caputo fractional derivative of order \(\alpha \in (1, 2]\), \(F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})\) is a multivalued map with compact, convex values (\(\mathcal{P}(\mathbb{R})\) is the family of all nonempty subsets of \(\mathbb{R}\)), \(y_0 \in \mathbb{R}\).

Fractional Differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, control, etc. (see [15, 17, 19, 18, 24, 27, 28]).

Recently, there has been a significant development in the study of ordinary and partial differential equations and inclusions involving fractional derivatives, see the monographs of Kilbas et al. [21], Lakshmikantham et al. [22], Miller and Ross [25], Podlubny [27], Samko et al. [29] and the papers by Agarwal et al. [1], Belarbi et al. [7, 8], Benchohra et al. [9, 10, 11, 12], Chang and Nieto [14], Diethelm et al. [15], and Ouahab [26].

Agarwal et al. [2] have considered a class of boundary value problems involving Riemann-Liouville fractional derivative on the half line. They used the diagonalization
process combined with the nonlinear alternative of Leray-Schauder type. This paper continues this study by considering a boundary value problem with the Caputo fractional derivative. We use the classical fixed point theorem of Bohnnenblust-Karlin [13] combined with the diagonalization process widely used for integer order differential equations; see for instance [3, 4]. Our results extend to the multivalued case those considered recently by Arara et al. [5].

2 Preliminary facts

We now introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $T > 0$ and $J := [0, T]$. $C(J, \mathbb{R})$ is the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the usual norm

$$\| y \| = \sup \{ |y(t)| : 0 \leq t \leq T \}.$$ 

$L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \rightarrow \mathbb{R}$ that are Lebesgue integrable with the norm

$$\| y \|_{L^1} = \int_0^T |y(t)| dt.$$ 

$AC^1(J, \mathbb{R})$ denote the space of differentiable functions whose first derivative $y'$ is absolutely continuous.

2.1 Fractional derivatives

**Definition 1.** ([21, 27]). Given an interval $[a,b]$ of $\mathbb{R}$. The fractional (arbitrary) order integral of the function $h \in L^1([a,b], \mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha_a h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where $\Gamma$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where $\delta$ is the delta function.

**Definition 2.** ([21]). For a given function $h$ on the interval $[a,b]$, the Caputo fractional-order derivative of $h$, is defined by

$$(^c D^\alpha_a h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(m)}(s) ds,$$

where $m = [\alpha] + 1$. 

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More details on fractional derivatives and their properties can be found in [21, 27]

**Lemma 3.** (Lemma 2.22 [21]). Let \( \alpha > 0 \), then the differential equation

\[
^{c}D^{\alpha}h(t) = 0
\]

has solutions

\[
h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{m-1} t^{m-1},
\]

for arbitrary \( c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, m - 1, \ m = \lfloor \alpha \rfloor + 1 \).

**Lemma 4.** (Lemma 2.22 [21]). Let \( \alpha > 0 \), then

\[
I^{\alpha}^{c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{m-1} t^{m-1},
\]

(2.1)

for arbitrary \( c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, m - 1, \ m = \lfloor \alpha \rfloor + 1 \).

### 2.2 Set-valued maps

Let \( X \) and \( Y \) be Banach spaces. A set-valued map \( G : X \to \mathcal{P}(Y) \) is said to be compact if \( G(X) = \bigcup \{ G(y) ; y \in X \} \) is compact. \( G \) has convex (closed, compact) values if \( G(y) \) is convex (closed, compact) for every \( y \in X \). \( G \) is bounded on bounded subsets of \( X \) if \( G(B) \) is bounded in \( Y \) for every bounded set \( B \) of \( X \). A set-valued map \( G \) is upper semicontinuous (usc for short) at \( z_0 \in X \) if for every open set \( O \) containing \( z_0 \) such that \( G(z_0) \subseteq O \). \( G \) is usc on \( X \) if it is usc at every point of \( X \) if \( G \) is nonempty and compact-valued then \( G \) is usc if and only if \( G \) has a closed graph. The set of all bounded closed convex and nonempty subsets of \( X \) is denoted by \( \mathcal{P}_{b,c}(X) \). A set-valued map \( G : J \to \mathcal{P}_{cl}(X) \) is measurable if for each \( y \in X \), the function \( t \mapsto dist(y, G(t)) \) is measurable on \( J \). If \( X \subseteq Y \), \( G \) has a fixed point if there exists \( y \in X \) such that \( y \in G(y) \). Also, \( \|G(y)\| = \sup\{|x| ; x \in G(y)\} \). A multivalued map \( G : J \to \mathcal{P}_{cl}(\mathbb{R}) \) is said to be measurable if for every \( y \in \mathbb{R} \), the function

\[
t \mapsto d(y, G(t)) = \inf\{|y - z| ; z \in G(t)|
\]

is measurable. For more details on multivalued maps see the books of Aubin and Frankowska [6], Deimling [16] and Hu and Papageorgiou [20].

**Definition 5.** A multivalued map \( F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is said to be \( L^{1}\)-Carathéodory if

(i) \( t \mapsto F(t, y) \) is measurable for each \( x \in \mathbb{R} \);

(ii) \( x \mapsto F(t, y) \) is upper semicontinuous for almost all \( t \in J \);

(iii) for each \( q > 0 \), there exists \( \varphi_q \in L^{1}(J, \mathbb{R}_+) \) such that

\[
\|F(t, x)\| \leq \varphi_q(t) \text{ for all } |x| \leq q \text{ and for a.e. } t \in J.
\]

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The multivalued map \( F \) is said to be Carathéodory if it satisfies (i) and (ii). For each \( y \in \mathcal{C}(J, \mathbb{R}) \), define the set of selections of \( F \) by
\[
S^1_{F,y} = \{ v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J \}.
\]

**Definition 6.** By a solution of BVP (1.1)-(1.2) we mean a function \( y \in AC^1(J, \mathbb{R}) \) such that
\[
\begin{align*}
^cD^\alpha y(t) &= g(t), \quad t \in J, \quad 1 < \alpha \leq 2, \\
y(0) &= y_0, \quad y \text{ bounded on } J,
\end{align*}
\]
where \( g \in S^1_{F,y} \).

**Remark 7.** Note that for an \( L^1 \)-Carathéodory multifunction \( F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R}) \), the set \( S^1_{F,y} \) is not empty (see [23]).

**Lemma 8.** (Bohnenblust-Karlin) (13). Let \( X \) be a Banach space and \( K \in \mathcal{P}_{cl,c}(X) \) and suppose that the operator \( G : K \to \mathcal{P}_{cl,c}(K) \) is upper semicontinuous and the set \( G(K) \) is relatively compact in \( X \). Then \( G \) has a fixed point in \( K \).

## 3 Main result

We first address a boundary value problem on a bounded domain. Let \( n \in \mathbb{N} \), and consider the boundary value problem
\[
\begin{align*}
^cD^\alpha y(t) &= F(t, y(t)), \quad t \in J_n := [0, n], \quad 1 < \alpha \leq 2, \\
y(0) &= y_0, \quad y'(n) = 0.
\end{align*}
\]

Let \( h : J_n \to \mathbb{R} \) be continuous, and consider the linear fractional order differential equation
\[
^cD^\alpha y(t) = h(t), \quad t \in J_n, \quad 1 < \alpha \leq 2.
\]

We shall refer to (3.1)-(3.2) as (LP).

By a solution to (LP) we mean a function \( y \in AC^1(J_n, \mathbb{R}) \) that satisfies equation (3.3) on \( J_n \) and condition (3.2).

We need the following auxiliary result:

**Lemma 9.** A function \( y \) is a solution of the fractional integral equation
\[
y(t) = y_0 + \int_0^t G_n(t, s)h(s)ds,
\]
if and only if \( y \) is a solution of (LP), where \( G(t, s) \) is the Green’s function defined by
\[
G_n(t, s) = \begin{cases} 
\frac{-t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq n, \\
\frac{-t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq t \leq s < n.
\end{cases}
\]
Proof. Let \( y \in C(J_n, \mathbb{R}) \) be a solution to (LP). Using Lemma 4, we have that
\[
y(t) = I^\alpha h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - c_0 - c_1 t,
\]
for arbitrary constants \( c_0 \) and \( c_1 \). We have by derivation
\[
y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds - c_1.
\]
Applying the boundary conditions (3.2), we find that
\[
c_0 = -y_0, \quad c_1 = \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds.
\]
Reciprocally, let \( y \in C(J_n, \mathbb{R}) \) satisfying (3.4), then
\[
y(t) = y_0 + \int_0^t \frac{-(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \int_0^n \frac{-(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds.
\]
Then \( y(0) = y_0 \) and
\[
y'(t) = \int_0^n \frac{-(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds.
\]
Thus, \( y'(n) = 0 \) and
\[
c D^{\alpha-1} y(t) = c D^{\alpha} y(t) = c D^{\alpha-1} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s) ds = c D^{\alpha-1} I^{\alpha-1} y(t).
\]

Remark 10. For each \( n > 0 \), the function \( t \in J_n \mapsto \int_0^n |G_n(t,s)| ds \) is continuous on \([0,n]\), and hence is bounded. Let
\[
\tilde{G}_n = \sup \left\{ \int_0^n |G_n(t,s)| ds, \ t \in J_n \right\}.
\]

Definition 11. A function \( y \in AC^1(J_n, \mathbb{R}) \) is said to be a solution of (3.1)–(3.2) if there exists a function \( v \in L^1(J_n, \mathbb{R}) \) with \( v(t) \in F(t, y(t)) \), for a.e. \( t \in J_n \), such that the differential equation \( c D^{\alpha} y(t) = v(t) \) on \( J_n \) and
\[
y(0) = y_0, \quad y'(n) = 0
\]
are satisfied.
Theorem 12. Assume the following hypotheses hold:

$(H_1)$ \( F : J_n \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is Carathéodory with compact convex values,

$(H_2)$ there exist \( p \in C(J, \mathbb{R}^+) \) and \( \psi : [0, \infty) \to (0, \infty) \) continuous and nondecreasing such that

\[
\|F(t, u)\|_P \leq p(t)\psi(|u|) \quad \text{for } t \in J_n \text{ and each } u \in \mathbb{R};
\]

$(H_3)$ There exists a constant \( r > 0 \) such that

\[
r \geq |y_0| + p_n^*\psi(r)\hat{G}_n,
\]

where

\[
p_n^* = \sup\{p(s), s \in J_n\}.
\]

Then BVP (3.1)–(3.2) has at least one solution on \( J_n \) with \( |y(t)| \leq r \) for each \( t \in J_n \).

Proof. Fix \( n \in \mathbb{N} \) and consider the boundary value problem

\[
D^\alpha y(t) \in F(t, y(t)), \quad t \in J_n, \quad 1 < \alpha \leq 2, \quad (3.6)
\]

\[
y(0) = y_0, \quad y'(n) = 0. \quad (3.7)
\]

We begin by showing that (3.6)-(3.7) has a solution \( y_n \in C(J_n, \mathbb{R}) \) with

\[
|y_n(t)| \leq r \quad \text{for each } t \in J_n.
\]

Consider the operator \( N : C(J_n, \mathbb{R}) \to 2^{C(J_n, \mathbb{R})} \) defined by

\[
(Ny) = \left\{ h \in C(J, \mathbb{R}) : h(t) = y_0 + \int_0^n G_n(t, s)v(s)ds \right\}
\]

where \( v \in S_{F,y}^1 \), and \( G_n(t, s) \) is the Green’s function given by (3.5). Clearly, from Lemma 8, the fixed points of \( N \) are solutions to (3.6)–(3.7). We shall show that \( N \) satisfies the assumptions of Bohnenblust-Karlin fixed point theorem. The proof will be given in several steps.

Let

\[
K = \{ y \in C(J_n, \mathbb{R}), \|y\|_n \leq r \},
\]

where \( r \) is the constant given by \( (H_3) \). It is clear that \( K \) is a closed, convex subset of \( C(J_n, \mathbb{R}) \).

**Step 1:** \( N(y) \) is convex for each \( y \in K \).
Indeed, if $h_1$, $h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}^1$ such that for each $t \in J_n$ we have

$$h_i(t) = y_0 + \int_0^t G_n(t,s)v_i(s)ds, \quad i = 1, 2.$$ 

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1 - d)h_2)(t) = \int_0^t G_n(t,s)(dv_1 + (1 - d)v_2(s)ds.$$ 

Since $S_{F,y}^1$ is convex (because $F$ has convex values), we have

$$dh_1 + (1 - d)h_2 \in N(y).$$

**Step 2:** $N(K)$ is bounded.

This is clear since $N(K) \subset K$ and $K$ is bounded.

**Step 3:** $N(K)$ is equicontinuous.

Let $\xi_1, \xi_2 \in J$, $\xi_1 < \xi_2$, $y \in K$ and $h \in N(y)$, then

$$|h(\xi_2) - h(\xi_1)| \leq \int_0^\xi |G(\xi_2, s) - G(\xi_1, s)|v(s)|ds \\ \leq p_n \psi(r) \int_0^\xi |G_n(\xi_2, s) - G_n(\xi_1, s)|ds.$$ 

As $\xi_1 \to \xi_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : K \longrightarrow \mathcal{P}(K)$ is compact.

**Step 4:** $N$ has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}^1$ such that, for each $t \in J_n$,

$$h_n(t) = y_0 + \int_0^t G_n(t,s)v_n(s)ds.$$ 

We must show that there exists $v_* \in S_{F,y_*}^1$ such that, for each $t \in J_n$,

$$h_*(t) = y_0 + \int_0^t G_n(t,s)v^*(s)ds.$$ 

We consider the continuous linear operator

$$\Gamma : L^1(J_n, \mathbb{R}) \rightarrow C(J_n, \mathbb{R}).$$
defined by

\[(\Gamma v)(t) = \int_0^n G_n(t, s)v(s)ds.\]

Since \(h_n(t) - y_0 \in \Gamma(S^1_{F,y})\), \(|(h_n(t) - y_0) - (h_\ast(t) - y_0)| \to 0\) as \(n \to \infty\) and \(\Gamma \circ S^1_F\) has a closed graph, then

\[h_\ast - y_0 \in \Gamma(S^1_{F,y}).\]

So

\[h_\ast \in N(y_\ast).\]

Therefore, we deduce from Bohnenblust-Karlin fixed point theorem that \(N\) has a fixed point \(y_n\) in \(K\) which is a solution of BVP \((3.6)-(3.7)\) with

\[|y_n(t)| \leq r \text{ for each } t \in J_n.\]

**Diagonalization process**

We now use the following diagonalization process. For \(k \in \mathbb{N}\), let

\[u_k(t) = \begin{cases} y_k(t), & t \in [0, n_k], \\ y_k(n_k) & t \in [n_k, \infty). \end{cases} \tag{3.8}\]

Here \(\{n_k\}_{k \in \mathbb{N}}\) is a sequence of numbers satisfying

\[0 < n_1 < n_2 < \ldots < n_k < \ldots \uparrow \infty.\]

Let \(S = \{u_k\}_{k=1}^{\infty}\). Notice that

\[|u_k(t)| \leq r \text{ for } t \in [0, n_1], \quad k \in \mathbb{N}.\]

Also for \(k \in \mathbb{N}\) and \(t \in [0, n_1]\) we have

\[u_{n_k}(t) = y_0 + \int_0^{n_1} G_{n_1}(t, s)v_{n_k}(s)ds,\]

where \(v_{n_k} \in S^1_{F,u_{n_k}}\) and thus, for \(k \in \mathbb{N}\) and \(t, x \in [0, n_1]\) we have

\[u_{n_k}(t) - u_{n_k}(x) = \int_0^{n_1} [G_{n_1}(t, s) - G_{n_1}(x, s)]v_{n_k}(s)ds\]

and by \((H_2)\), we have

\[|u_{n_k}(t) - u_{n_k}(x)| \leq p^*_r(r) \int_0^{n_1} |G_{n_1}(t, s) - G_{n_1}(x, s)|ds.\]
The Arzelà-Ascoli Theorem guarantees that there is a subsequence $N^*_1$ of $\mathbb{N}$ and a function $z_1 \in C([0, n_1], \mathbb{R})$ with $u_{n_k} \to z_1$ in $C([0, n_1], \mathbb{R})$ as $k \to \infty$ through $N^*_1$. Let $N_1 = N^*_1 \setminus \{1\}$. Notice that

$$|u_{n_k}(t)| \leq r \text{ for } t \in [0, n_2], k \in \mathbb{N}.$$ 

Also for $k \in \mathbb{N}$ and $t, x \in [0, n_2]$ we have

$$|u_{n_k}(t) - u_{n_k}(x)| \leq p^*_2 \psi(r) \int_0^{n_2} |G_{n_2}(t, s) - G_{n_2}(x, s)| ds.$$ 

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $N^*_2$ of $N_1$ and a function $z_2 \in C([0, n_2], \mathbb{R})$ with $u_{n_k} \to z_2$ in $C([0, n_2], \mathbb{R})$ as $k \to \infty$ through $N^*_2$. Note that $z_1 = z_2$ on $[0, n_1]$ since $N^*_2 \subseteq N_1$. Let $N_2 = N^*_2 \setminus \{2\}$. Proceed inductively to obtain for $m \in \{3, 4, \ldots\}$ a subsequence $N^*_m$ of $N_{m-1}$ and a function $z_m \in C([0, n_m], \mathbb{R})$ with $u_{n_k} \to z_m$ in $C([0, n_m], \mathbb{R})$ as $k \to \infty$ through $N^*_m$. Let $N_m = N^*_m \setminus \{m\}$.

Define a function $y$ as follows. Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \leq n_m$. Then define $y(t) = z_m(t)$. Then $y \in C([0, \infty), \mathbb{R})$, $y(0) = y_0$ and $|y(t)| \leq r$ for $t \in [0, \infty)$.

Again fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \leq n_m$. Then for $n \in N_m$ we have

$$u_{n_k}(t) = y_0 + \int_0^{n_m} G_{n_m}(t, s) v_{n_k}(s) ds,$$

Let $n_k \to \infty$ through $N_m$ to obtain

$$z_m(t) = y_0 + \int_0^{n_m} G_{n_m}(x, s) v_m(s) ds,$$

i.e

$$y(t) = y_0 + \int_0^{n_m} G_{n_m}(t, s) v(s) ds,$$

where $v_m \in S_{F, z_m}^1$.

We can use this method for each $x \in [0, n_m]$, and for each $m \in \mathbb{N}$. Thus

$$D^n y(t) \in F(t, y(t)), \text{ for } t \in [0, n_m]$$

for each $m \in \mathbb{N}$ and $\alpha \in (1, 2]$.

\[\square\]

4  An example

Consider the boundary value problem

$$^cD^n y(t) \in F(t, y(t)), \text{ for } t \in J = [0, \infty), \quad 1 < \alpha \leq 2, \quad (4.1)$$

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where \( cD^{\alpha} \) is the Caputo fractional derivative. Set
\[
F(t,y) = \{ v \in \mathbb{R} : f_1(t,y) \leq v \leq f_2(t,y) \},
\]
where \( f_1, f_2 : J \times \mathbb{R} \to \mathbb{R} \) are measurable in \( t \). We assume that for each \( t \in J \), \( f_1(t,\cdot) \) is lower semi-continuous (i.e, the set \( \{ y \in \mathbb{R} : f_1(t,y) > \mu \} \) is open for each \( \mu \in \mathbb{R} \)), and assume that for each \( t \in J \), \( f_2(t,\cdot) \) is upper semi-continuous (i.e the set \( \{ y \in \mathbb{R} : f_2(t,y) < \mu \} \) is open for each \( \mu \in \mathbb{R} \)). Assume that there exists \( p \in C(J,\mathbb{R}^+) \) and \( \delta \in (0,1) \) such that
\[
\max(|f_1(t,y)|, |f_2(t,y)|) \leq p(t)|y|^\delta, \quad t \in J, \text{ and all } y \in \mathbb{R}.
\]
It is clear that \( F \) is compact and convex valued, and it is upper semi-continuous (see [16]). Also conditions (\( \mathcal{H}_1 \)) and (\( \mathcal{H}_2 \)) are satisfied with
\[
\psi(u) = u^\delta, \quad \text{for each } u \in [0,\infty).
\]
From (3.5) we have for \( s \leq t \)
\[
\int_0^t G_n(t,s)ds = \frac{t}{\Gamma(\alpha-1)(\alpha-1)}[(n-t)^{(\alpha-1)} - n^{(\alpha-1)}] + \frac{t^\alpha}{\alpha\Gamma(\alpha)}
\]
and for \( t \leq s \)
\[
\int_t^s G_n(t,s)ds = -\frac{t}{(\alpha-1)\Gamma(\alpha-1)}(n-t)^{\alpha-1}.
\]
Also since
\[
\lim_{c \to \infty} \frac{c}{1 + p_n^* \psi(c)G_n} = \lim_{c \to \infty} \frac{c}{\psi(c)} = \lim_{c \to \infty} \frac{c}{c^\delta} = \infty,
\]
then there exists \( r > 0 \) such that
\[
\frac{r}{1 + p_n^* \psi(r)G_n} \geq 1.
\]
Hence (\( \mathcal{H}_3 \)) is satisfied. Then by Theorem 12, BVP (4.1)-(4.2) has a bounded solution on \([0,\infty)\).

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**References**


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