FIXED POINTS OF MAPPINGS WITH
DIMINISHING PROBABILISTIC ORBITAL
DIAMETERS

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Abstract. In this paper we prove a fixed point theorem for a pair of mappings with probabilis-
tic diminishing orbital diameters on Menger spaces and introduce the notion of generalized joint
diminishing probabilistic orbital diameters (gjdpod) for a quadruplet of mappings.

1 Introduction

The notion of ‘diminishing orbital diameters’ (dod) was introduced by Belluce and
Kirk [1]. Subsequently, Fisher [3], Huang, Huang and Jeng [4], Liu [7], Ranganathan,
Srivastva and Gupta [9], Singh [10], Wong [12] etc. obtained some more results in
this settings.
Istrătescu and Săcuiu [5] introduced the concept of non-expansive mappings and
mapping with ‘diminishing probabilistic orbital diameters’ (dpod) on probabilistic
metric spaces (PM-spaces). Singh and Pant [11] have shown that a non-expansive
mapping on PM-space having dpod has a fixed point. They have also investigated
that the condition of non-expansiveness of the mapping may be relaxed to the con-
dition of the mapping being with relatively compact orbits.
In this paper we introduce the notion of dpod and gjdpod for a pair of mappings
and established a fixed point theorem. Subsequently, we introduced the concept of
gjdpod for a quadruplet of mappings and prove a fixed point theorem. Some of the
previously results of [7], [9], [10], [11] (in different settings) may be derived from our
results.

1.1 Preliminaries

Definition 1. [2].Let $A$ be a non-empty subset of $X$. The function $D_A(\cdot)$ defined
by

$$D_A(x) = \sup_{\varepsilon < x} \inf_{u,v \in A} F_{u,v}(\varepsilon)$$


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is called the probabilistic diameter of $A$.

**Definition 2.** [2] The function $E_{A,B}(\cdot)$ defined by

$$E_{A,B}(\varepsilon) = \text{lub} \{ \text{glb} \{ \text{lub} F_{u,v}(x) \} : u \in A, v \in B \}$$

is called the probabilistic distance between $A$ and $B$.

Let $P : X \rightarrow X$ and $u \in X$, then $O_P(u) = (u, Pu, P^2u, \ldots)$ is called the orbit of $u$ with respect to $P$ and $\overline{O_P(u)}$ denotes the closure of $O_P(u)$.

**Definition 3.** [5] Let $P$ be a self map on a PM-space $X$. $P$ is said to have dpod at $u$ if for $D_{O_P(P(u))}(\varepsilon) > 0$

$$\lim_{n \to \infty} D_{O_P(P^0(u))}(\varepsilon) > D_{O_P(u)}(\varepsilon)$$

where $H$ is a distribution function.

We now introduce the following definitions:

**Definition 4.** A pair of mappings $P,Q$ of a PM-space $X$ is said to have diminishing probabilistic orbital diameters (dpod) if

$$\lim_{n \to \infty} E_{O_P(P^n(u)),O_Q(Q^n(u))}(\varepsilon) > E_{O_P(u),O_Q(u)}(\varepsilon), \varepsilon > 0,$$

for all $u \in X$ with $E_{O_P(P^n(u)),O_Q(Q^n(u))}(\varepsilon) \neq H$.

**Definition 5.** A pair of mappings $P,Q$ on a PM-space $X$ is said to have generalized diminishing probabilistic orbital diameters (gdpod) if

$$\lim_{n \to \infty} E_{O_P(P^n(u)),O_Q(Q^n(v))}(\varepsilon) > E_{O_P(u),O_Q(v)}(\varepsilon), \varepsilon > 0,$$

for all $u \in X$ with $E_{O_P(P^n(u)),O_Q(Q^n(v))}(\varepsilon) \neq H$.

It is clear that $(P,Q)$ has a dpod if $(P,Q)$ has a gdpod. Also $(P,P)$ has a dpod if and only if $P$ has dpod.
2 Main Result

**Theorem 6.** Let $P$ and $Q$ are continuous self mappings of a compact Menger space. If the pair $(P,Q)$ has a gdpod on $X$, then for each $u,v \in X$, there exists some subsequences $\{P^n(u)\}$ of $P_n(u)$ and $\{Q^n(u)\}$ of $Q_n(u)$ converge to a common fixed point for $P$ and $Q$.

**Proof.** Let $u \in X$, $L(u)$ denotes the set of all points of $X$ which are the limits of the subsequence $P_n(u)$ Since $L(u) \neq \emptyset$ because $X$ is compact, $L(u)$ is mapped into itself by $P$. Also $L(u)$ is closed, so by Zorn’s lemma there exists a minimal $P$-invariant non-empty subset $A \subset L(u)$ such that $A$ is closed and mapped into itself by $P$. Similarly we can find a minimal $Q$-invariant non-empty subset $B \subset L(v)$ such that $B$ is closed and mapped into itself by $Q$. For $u_0 \in A$, $O_P(u_0)$ is mapped into itself by $P$. Therefore minimality of $A$ implies that $A = O_P(u_0)$. Similarly for $v_0 \in B$, we have $B = O_Q(v_0)$.

We now prove that $E_{A,B}(\varepsilon) = H$, $\varepsilon > 0$. Suppose $E_{A,B}(\varepsilon) \neq H$, $\varepsilon > 0$. Since $P,Q$ has a dpod, we have

$$E_{A,B}(\varepsilon) = E_{O_P(u_0),O_Q(v_0)}(\varepsilon) < \lim_{n \to \infty} E_{O_P(P_n(u_0)),O_Q(Q_n(v_0))}(\varepsilon)$$

This implies

$$E_{O_P(u_0),O_Q(v_0)}(\varepsilon) < \lim_{n \to \infty} E_{O_P(P_n(u_0)),O_Q(Q_n(v_0))}(\varepsilon) = E_{A,B}(\varepsilon)$$

contradiction. Hence $E_{A,B}(\varepsilon) = H$, which implies that $A = B = (w)$ (say). Then it is clear that $w$ is a common fixed point of $P$ and $Q$. If $z$ is another fixed point of $P$ and $z \neq w$ Then we have

$$\lim_{n \to \infty} E_{O_P(P_n(z)),O_Q(Q_n(w))}(\varepsilon) > E_{O_P(z),O_Q(w)}(\varepsilon), \varepsilon > 0, \quad \text{or } E_{z,w}(\varepsilon) > E_{z,w}(\varepsilon),$$

a contradiction. Hence $w$ is a unique fixed point of $P$. Similarly, we may show that $w$ is a unique fixed point of $Q$.

This completes the proof of the theorem.

**Remark 7.** If in the above theorem condition gdpod is replaced by the condition dpod, then it no longer assures the existence of a common fixed point for $P$ and $Q$. (see [7])

**Corollary 8.** Let $P$ be a continuous self mapping of a compact Menger space $X$. If $(P,P)$ has a gdpod, then $P$ has a unique fixed point. Furthermore, for each $u \in X$, there exists some subsequences of $P^n(u)$ converge to a unique fixed point of $P$.

Pant, Dimri and Chandola [8] have introduced the concept of joint sequence of iterates for a quadruplet of mappings as follows:

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Definition 9. [8] Let $B = (P,Q,S,T)$ be a quadruplet of self mappings on a PM-space $X$. For $u_0$ in $X$, let $Tu_n = QPu_{n-1}$, if $n$ is odd and $Tu_n = SQPu_{n-1}$ if $n$ is even, then the sequence

$$J_B(u_0) = \{u_0, Pu_0, QPu_0, SQPu_0, TSQPu_0, \cdots\}$$

is called the joint sequence of iterates of $B$ at $u_0$.

We now introduce the notion of gjdpod for a quadruplet of mappings in PM-space.

Let $\delta_u(\varepsilon) = \lim\limits_{n\to\infty} D_{J_B^n(u)}(\varepsilon)$.

Definition 10. $B$ will be called to have gjdpod at $u$ if for $D_{J_B^n(u)}(\varepsilon) \neq H, \varepsilon > 0$,

$$\delta_u(\varepsilon) > D_{J_B(u)}(\varepsilon).$$

Theorem 11. Let $X$ be a compact Menger space and $B = (P,Q,S,T)$ be a quadruplet of continuous self mappings on $X$ such that $B$ have gjdpod on $X$. Then for each $u_0 \in X$, a subsequence of $J_B(u_0)$ converges to a common fixed point of $P,Q,S$ and $T$.

Proof. For $u_0 \in X$, let $A(u_0)$ denote the set of all points of $X$ which are limit of subsequences of the sequence $J_B(u_0)$. Since $X$ is compact, $A(u_0) \neq \phi$ Also $A(u_0)$ is closed and mapped into itself by $P,Q,S$ and $T$. Let some subsequence of $J_B(u_0)$ converge to a point $u$ in $X$, so $u \in A(u_0)$. Further $P,Q,S$ and $T$ are continuous, therefore $J_B(u_0) \subset A(u_0)$. By Zorn’s Lemma, there exists a minimal nonempty subset $K \subset A(u_0)$ such that $K$ is closed and mapped into itself by $P,Q,S$ and $T$. Also for $q_0 \in K$, $J_B(q_0)$ is mapped into itself by $P,Q,S$ and $T$. Therefore minimality of $K$ implies that $K = J_B(q_0)$. Suppose $D_K(\varepsilon) \neq H, \varepsilon > 0$. Since $B$ have gjdpod, then we have

$$\delta_{q_0}(\varepsilon) > D_{J_B(q_0)}(\varepsilon).$$

This implies that $D_{J_{B,n}(q_0)}(\varepsilon) > D_{J_B(q_0)}(\varepsilon)$, for some integer $n$. Thus

$$D_{J_{B,n}(q_0)}(\varepsilon) > D_{J_B(q_0)}(\varepsilon), \varepsilon > 0$$

This shows that $J_{B,n}(q_0)$ is a proper subset of $K$, contradicting the minimality of $K$. Hence $D_K(\varepsilon) = H, \varepsilon > 0$ Thus $K$ consists of a single point $q_0$. So we have $P(q_0) = Q(q_0) = R(q_0) = S(q_0) = q_0$. Therefore $q_0$ is the common fixed point of $P,Q,S$ and $T$.

Remark 12. With $Q = S = T = I$ (Identity mapping), the notion of gjdpod is same as dpod and then result of Kirk (Th. A, [6]) follow.
Remark 13. If any two of $P, Q, S, T$ are taken as identity maps then $\text{gdpo}_{\text{d}}$ reduces to $\text{jdpo}_{\text{d}}$ and the result of Singh and Pant (Th. 4, [11]) is obtained as corollary.

Remark 14. It is not necessary that any continuous mapping $P$ in Theorem 11 has $\text{dpo}_{\text{d}}$ on $X$, since in such a case it might be possible to obtain a family $B$ of continuous self mappings on $X$ such that $B \cup P$ has a $\text{gdpo}_{\text{d}}$ (see, for illustration [9]).

References


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