A UNIQUE COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS

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Abstract. The aim of this paper is to establish a unique common fixed point theorem for two pairs of occasionally weakly compatible single and multi-valued maps in a metric space. This result improves the result of Türkoğlu et al. [6] and references therein.

1 Introduction and preliminaries

Throughout this paper, \((\mathcal{X}, d)\) denotes a metric space and \(CB(\mathcal{X})\) the family of all nonempty closed and bounded subsets of \(\mathcal{X}\). Let \(H\) be the Hausdorff metric on \(CB(\mathcal{X})\) induced by the metric \(d\); i.e.,

\[
H(A, B) = \max \{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \}
\]

for \(A, B\) in \(CB(\mathcal{X})\), where

\[
d(x, A) = \inf \{ d(x, y) : y \in A \}.
\]

Let \(f, g\) be two self-maps of a metric space \((X, d)\). In his paper [5], Sessa defined \(f\) and \(g\) to be weakly commuting if for all \(x \in \mathcal{X}\)

\[
d(fgx, gfx) \leq d(gx, fx).
\]

It can be seen that two commuting maps \((fgx = gfx \ \forall x \in \mathcal{X})\) are weakly commuting, but the converse is false in general (see [5]).

Afterwards, Jungck [2] extended the concepts of commutativity and weak commutativity by giving the notion of compatibility. Maps \(f\) and \(g\) above are compatible if

\[
\lim_{n \to \infty} d(fgx_n, gfx_n) = 0
\]

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whenever \( \{x_n\} \) is a sequence in \( \mathcal{X} \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in \mathcal{X} \). Obviously, weakly commuting maps are compatible, but the converse is not true in general (see [2]).

Further, Kaneko and Sessa [4] extended the concept of compatibility for single valued maps to the setting of single and multi-valued maps as follows: \( f : \mathcal{X} \to \mathcal{X} \) and \( F : \mathcal{X} \to CB(\mathcal{X}) \) are said to be compatible if

\[
\lim_{n \to \infty} H(Ffx_n, fFx_n) = 0,
\]

whenever \( \{x_n\} \) is a sequence in \( \mathcal{X} \) such that \( Fx_n \to A \in CB(\mathcal{X}) \) and \( fx_n \to t \in A \).

In 2002, Türkoğlu et al. [6] gave another generalization of commutativity and weak commutativity for single valued maps by introducing the next definition: \( f : \mathcal{X} \to \mathcal{X} \) and \( F : \mathcal{X} \to CB(\mathcal{X}) \) are said to be compatible if

\[
\lim_{n \to \infty} d(fy_n, Ffx_n) = 0
\]

whenever \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( \mathcal{X} \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = t \) for some \( t \in \mathcal{X} \), where \( y_n \in Fx_n \) for \( n = 1, 2, \ldots \).

In [3], Jungck and Rhoades weakened the notion of compatibility for single and multi-valued maps by giving the concept of weak compatibility. They define maps \( f \) and \( F \) above to be weakly compatible if \( fF x = F fx \) whenever \( fx \in Fx \).

Recently, Abbas and Rhoades [1] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weak compatibility (owc). Maps \( f \) and \( F \) are said to be owc if and only if there exists some point \( x \) in \( \mathcal{X} \) such that

\[ fx \in Fx \text{ and } fFx \subseteq Ffx. \]

For our main results we need the following lemma which whose proof is obvious.

**Lemma 1.** let \( A, B \) in \( CB(\mathcal{X}) \), then for any \( a \in A \) we have

\[ d(a, B) \leq H(A, B). \]

In their paper [6], Türkoğlu et al. proved the next result.

**Theorem 2.** Let \( (\mathcal{X}, d) \) be a complete metric space. Let \( f, g : \mathcal{X} \to \mathcal{X} \) be continuous maps and \( S, T : \mathcal{X} \to CB(\mathcal{X}) \) be \( H \)-continuous maps such that \( T(\mathcal{X}) \subseteq f(\mathcal{X}) \) and \( S(\mathcal{X}) \subseteq g(\mathcal{X}) \), the pair \( S \) and \( g \) are compatible maps and

\[
H^p(Sx, Ty) \leq \max\{ad(fx, gy)d^{p-1}(fx, Sx), ad(fx, gy)d^{p-1}(gy, Ty), \\
ad(fx, Sx)d^{p-1}(gy, Ty), cd^{p-1}(fx, Ty)d(gy, Sx)\}
\]

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for all \(x, y \in \mathcal{X}\), where \(p \geq 2\) is an integer, \(0 < a < 1\) and \(c \geq 0\). Then there exists a point \(z \in \mathcal{X}\) such that \(fz \in Sz\) and \(gz \in Tz\), i.e., \(z\) is a coincidence point of \(f, S\) and of \(g, T\). Further, \(z\) is unique when \(0 < c < 1\).

Our aim here is to establish and prove a unique common fixed point theorem by dropping the hypothesis of continuity required on the four maps in the above result, and deleting the two conditions \(T(\mathcal{X}) \subseteq f(\mathcal{X})\) and \(S(\mathcal{X}) \subseteq g(\mathcal{X})\) with \(a \geq 0\) in a metric space instead of a complete metric space, by using the concept of occasionally weakly compatible maps given in [6].

## 2 Main results

**Theorem 3.** Let \((\mathcal{X}, d)\) be a metric space. Let \(f, g : \mathcal{X} \to \mathcal{X}\) and \(F, G : \mathcal{X} \to CB(\mathcal{X})\) be single and multi-valued maps, respectively such that the pairs \(\{f, F\}\) and \(\{g, G\}\) are owc and satisfy inequality

\[
H^p(Fx, Gy) \leq \max\{ad(fx, gy)d^{p-1}(fx, Fx), ad(fx, gy)d^{p-1}(gy, Gy), ad(fx, Fx)d^{p-1}(gy, Gy), cH^{p-1}(fx, Gy)d(gy, Fx)\}
\]

for all \(x, y \in \mathcal{X}\), where \(p \geq 2\) is an integer, \(a \geq 0, 0 < c < 1\). Then \(f, g, F\) and \(G\) have a unique common fixed point in \(\mathcal{X}\).

**Proof.** Since the pairs \(\{f, F\}\) and \(\{g, G\}\) are owc, then there exist two elements \(u\) and \(v\) in \(\mathcal{X}\) such that \(fu \in Fu, fFu \subseteq Ffu\) and \(gv \in Gv, gGv \subseteq Ggv\).

First we prove that \(fu = gv\). By Lemma 1 and the triangle inequality we have \(d(fu, gv) \leq H(Fu, Gv)\). Suppose that \(H(Fu, Gv) > 0\). Then, by inequality (2.1) we get

\[
H^p(Fu, Gv) \leq \max\{ad(fu, gv)d^{p-1}(fu, Fu), ad(fu, gv)d^{p-1}(gv, Gv), ad(fu, Fu)d^{p-1}(gv, Gv), cH^{p-1}(fu, Gv)d(gv, Fu)\}
\]

Since \(d(fu, Gv) \leq H(Fu, Gv)\) and \(d(gv, Fu) \leq H(Fu, Gv)\) by Lemma 1, and then

\[
H^p(Fu, Gv) \leq cH^{p-1}(fu, Gv)d(gv, Fu) \leq cH^p(Fu, Gv) < H^p(Fu, Gv)
\]

which is a contradiction. Hence \(H(Fu, Gv) = 0\) which implies that \(fu = gv\).

Again by Lemma 1 and the triangle inequality we have

\[
d(f^2u, Fu) = d(ffu, gv) \leq H(Ffu, Gv).
\]

We claim that \(f^2u = fu\). Suppose not. Then \(H(Ffu, Gv) > 0\) and using inequality (2.1) we obtain

\[
H^p(Ffu, Gv) \leq \max\{ad(f^2u, gv)d^{p-1}(f^2u, Ffu), ad(f^2u, gv)d^{p-1}(gv, Gv), ad(f^2u, Ffu)d^{p-1}(gv, Gv), cH^{p-1}(f^2u, Gv)d(gv, Ffu)\}
\]

\[
= cH^{p-1}(f^2u, Gv)d(gv, Ffu).
\]
But $d(f^2u, Gv) \leq H((fu, Gv))$ and $d(gv, Ffu) \leq H((fu, Gv))$ by Lemma 1 and so

$$H^p(ffu, Gv) \leq cH^p(ffu, Gv) < H^p(ffu, Gv),$$

a contradiction. This implies that $H(ffu, Gv) = 0$, thus $f^2u = fu = gv$.

Similarly, we can prove that $g^2v = gv$.

Putting $fu = gv = z$, then, $fz = z = gz$, $z \in Fz$ and $z \in Gz$. Therefore $z$ is a common fixed point of maps $f, g, F$ and $G$.

Now, suppose that $f, g, F$ and $G$ have another common fixed point $z' \neq z$. Then, by Lemma 1 and the triangle inequality we have

$$d(z, z') = d(fz, gz') \leq H(Fz, Gz').$$

Assume that $H(Fz, Gz') > 0$. Then the use of inequality (2.1) gives

$$H^p(Fz, Gz') \leq \max\{ad(fz, gz')d^{p-1}(fz, Fz), ad(fz, gz')d^{p-1}(gz', Gz'),
\quad ad(fz, Fz)d^{p-1}(gz', Fz), cd^{p-1}(fz, Gz')d(gz', Fz)\}$$

$$= cd^{p-1}(fz, Gz')d(gz', Fz).$$

Then since $d(fz, Gz') \leq H(Fz, Gz')$ and $d(gz', Fz) \leq H(Fz, Gz')$, we have

$$H^p(Fz, Gz') \leq cH^p(Fz, Gz') < H^p(Fz, Gz'),$$

a contradiction. Then $H(Fz, Gz') = 0$ and hence $z' = z$. \hfill \Box

If we put in Theorem 3 $f = g$ and $F = G$, we obtain the following result.

**Corollary 4.** Let $(X, d)$ be a metric space and let $f : X \rightarrow X$, $F : X \rightarrow CB(X)$ be a single and a multi-valued map respectively. Suppose that $f$ and $F$ are owc and satisfy the inequality

$$H^p(Fx, Fy) \leq \max\{ad(fx, fy)d^{p-1}(fx, Fx), ad(fx, fy)d^{p-1}(fy, Fy),
\quad ad(fx, Fx)d^{p-1}(fy, Fy), cd^{p-1}(fx, Fy)d(fy, Fx)\}$$

for all $x, y$ in $X$, where $p \geq 2$ is an integer, $a \geq 0$ and $0 < c < 1$. Then, $f$ and $F$ have a unique common fixed point in $X$.

Now, letting $f = g$ we get the next corollary.

**Corollary 5.** Let $(X, d)$ be a metric space, $f : X \rightarrow X$ be a single map and $F, G : X \rightarrow CB(X)$ be two multi-valued maps such that

(i) the pairs $\{f, F\}$ and $\{f, G\}$ are owc,

(ii) the inequality

$$H^p(Fx, Gy) \leq \max\{ad(fx, fy)d^{p-1}(fx, Fx), ad(fx, fy)d^{p-1}(fy, Gy),
\quad ad(fx, Fx)d^{p-1}(fy, Gy), cd^{p-1}(fx, Gy)d(fy, Fx)\}$$

holds for all $x, y$ in $X$, where $p \geq 2$ is an integer, $a \geq 0$ and $0 < c < 1$. Then, $f, F$ and $G$ have a unique common fixed point in $X$.

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Now, we give an example which illustrate our main result.

**Example 6.** Let \( X = [0,2] \) endowed with the Euclidean metric \( d \). Define \( f, g : X \to X \) and \( F, G : X \to CB(X) \) as follows:

\[
f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 < x \leq 2, \end{cases} \quad F(x) = \begin{cases} \{1\} & \text{if } 0 \leq x \leq 1 \\ \{0\} & \text{if } 1 < x \leq 2, \end{cases}
\]

\[
g(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 < x \leq 2, \end{cases} \quad G(x) = \begin{cases} \{1\} & \text{if } 0 \leq x \leq 1 \\ \{\frac{x}{2}\} & \text{if } 1 < x \leq 2. \end{cases}
\]

First, we have

\[
f(1) = 1 \in F(1) = \{1\} \quad \text{and} \quad fF(1) = \{1\} = Ff(1)
\]

and

\[
g(1) = 1 \in G(1) = \{1\} \quad \text{and} \quad gG(1) = \{1\} = Gg(1);
\]

i.e., \( f \) and \( F \) as well as \( g \) and \( G \) are owc.

Also, for all \( x \) and \( y \) in \( X \), inequality (2.1) is satisfied for a large enough \( a \).

So, all hypotheses of Theorem 3 are satisfied and \( 1 \) is the unique common fixed point of \( f, g, F \) and \( G \).

On the other hand, it is clear to see that maps \( f, g, F \) and \( G \) are discontinuous at \( t = 1 \).

Further, we have

\[
F(X) = \{0,1\} \subset f(X) = [0,1] \cup \{2\} \quad \text{but} \quad G(X) = \left[\frac{1}{2},1\right] \not\subset g(X) = \{1,2\}.
\]

So, this example illustrate the generality of our result.

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