FAMILIES OF QUASI-PSEUDO-METRICS GENERATED BY PROBABILISTIC QUASI-PSEUDO-METRIC SPACES

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Abstract. This paper contains a study of families of quasi-pseudo-metrics (the concept of a quasi-pseudo-metric was introduced by Wilson [22], Albert [1] and Kelly [9]) generated by probabilistic quasi-pseudo-metric-spaces which are generalization of probabilistic metric space (PM-space shortly) [2, 3, 4, 6]. The idea of PM-spaces was introduced by Menger [11, 12], Schweizer and Sklar [18] and Serstnev [19]. Families of pseudo-metrics generated by PM-spaces and those generalizing PM-spaces have been described by Stevens [20] and Nishiure [14].

1 Introduction

The concept of a probabilistic metric space is a generalization of a metric spaces. The origin of the theory data back to a paper published by Menger in 1942 [11]. A foundational paper on the subject was written by Schweizer and Sklar in [16, 17] and numerous articles follows thereafter. The latter two authors gave an excellent treatment of the subject in their book published in 1983 [18].

The concept of a quasi-metric space (where the condition of symmetry in dropped) was introduced in Wilson [22] and further developed in Kelly [9].

In the development of the theory of quasi-pseudo-metric spaces two streams can be distinguished. The core of the first is the concept of a convergent sequence (see [Kelly [9]]). The second stream, a structure topological one, connected with Kelly as well, originated from the observation that every quasi-pseudo-metric on a given set does naturally generate a dual quasi-pseudo-metric on the same set. Thus a system of two mutually conjugates functions appeared. The dropped symmetry condition thus manifested itself in an external nature of such systems. Since each quasi-pseudo-metric generates a topology, hence of systems of two topologies can be associated with every quasi-pseudo-metric (Kelly [9]).

2000 Mathematics Subject Classification: 54E40.

Keywords: Families generated by $P_{pq}M$-spaces; Quasi-pseudo-Menger space; Probabilistic quasi-pseudo-metric spaces ($P_{pq}M$-space); Statistical quasi-metric space ($S_{pq}M$-space).

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The purpose of this study is to invalidate a natural generalization of probabilistic metric space and quasi-pseudo-metric space (Birsan [2, 3, 4], Grabiec [6]).

This paper contains a study of families of quasi-pseudo-metrics generated by Probabilistic-quasi-pseudo-metric-spaces which are generalization of probabilistic metric spaces (PM-spaces) ([2, 3, 4, 6]). The idea of PM-spaces goes back to Menger [11], [12]. The families of pseudo-metrics generated by PM-spaces and these generalizing PM-spaces have been described by Stevens [20] and Nishiura [14].

2 Preliminaries

A distance distribution function (d.d.f.) is a non-decreasing function $F : [0, +\infty] \rightarrow [0, 1]$, which is left-continuous on $(0, +\infty)$, and assumes the values $F(0) = 0$ and $F(+\infty) = 1$. The set of all d.d.f’s, denoted by $\Delta^+$, is equipped with modified Le\'vy metric $d_L$ (see pp. 45 of [18]). The metric space $(\Delta^+, d_L)$ is compact and hence complete. Further, $\Delta^+$ is partially ordered by usual order for real-valued functions.

Let $u_a$ be the element of $\Delta^+$ defined by

$$u_a = \begin{cases} 1_{(a, +\infty]}, & \text{for all } a \in [0, +\infty), \\ 1_{(+\infty)}, & \text{for } a = \{+\infty\}. \end{cases}$$

A triangle function $*$ is defined to be a binary operation on $\Delta^+$ which is non-decreasing in each component, and if $(\Delta^+, *)$ is an Abelian monoid with the identity $u_0$.

Triangle functions considered in this paper will be assumed to be continuous with respect to the topology induced by metric $d_L$.

Definition 1. Let $p_L : \Delta^+ \times \Delta^+ \rightarrow I$ be defined by the following formula:

$$p_L(F, G) = \inf \{ h \in (0, 1] : G(t) \leq F(t + h) + h, \ t \in (0, \frac{1}{h}) \}. \quad (1)$$

Observe that, for all $F, G \in \Delta^+$, we have $G(t) \leq F(t + 1) + 1$. Hence the set of (1) is nonempty.

Lemma 2. If $p_L(F, G) = h > 0$, then, for every $t \in (0, \frac{1}{h})$, $G(t) \leq F(t + h) + h$.

Proof. For arbitrary $s > 0$ let $J_s = (0, \frac{1}{s})$. Then $J_{s_2} \subseteq J_{s_1}$ whenever $0 < s_1 < s_2 < 1$. Let $t \in J_h$. Since the interval $J_h$ is open, there exist $t_1 < t$ and $s > 0$ such that $t_1 \in J_{h+s}$. As $p_L(F, G) = h$, we get $G(t_1) \leq F(t_1 + h + s) + (h + s)$. Let $s \rightarrow 0$. Then $G(t_1) \leq F(t + h) + h$ since $F$ is nondecreasing.

Next, let $t_1 \rightarrow t$. Using the left-continuity of $G$, we obtain $G(t) \leq F(t + h) + h$ for $t \in J_h$. This completes the proof. \qed
Theorem 3. The function \( p_L : \Delta^+ \times \Delta^+ \to I \) defined by (1) is a quasi-pseudo-metric on \( \Delta^+ \). Recall that a quasi-pseudo-metric space is an ordered pair \((X, p)\), where \( X \) is a nonempty set and the function \( p : X^2 \to R^+ \) satisfies the following conditions: for all \( x, y, z \in X \),
\[
\begin{align*}
d(x, x) &= 0, \\
d(x, y) &\leq d(x, z) + d(z, y).
\end{align*}
\]

Proof. For each \( F \in \Delta^+ \) we have \( p_L(F, F) = 0 \). This is the direct consequence of Definition 1. In order to prove the “triangle inequality”:
\[
p_L(F, H) \leq p_L(F, G) + p_L(G, H) \quad \text{for} \quad F, G, H \in \Delta^+,
\]
Let \( x = p_L(F, G) > 0 \) and \( y = p_L(G, H) > 0 \). If \( x + y \geq 1 \), then (1) is satisfied. Thus let \( x + y < 1 \) and \( t \in J_{x+y} \). Then \( t + y \in J_x \). Using this fact and Lemma 2, we obtain
\[
H(t) \leq F(t+(x+y)) + (x+y) \leq F(t + y + x) + y + y.
\]
Thus the equality \( H(t) \leq F(t+(x+y)) + (x+y) \) holds for \( t \in J_{x+y} \). Consequently, we have \( p_L(F, H) \leq x + y = p_L(F, G) + p_L(G, H) \).

The definition of the quasi-pseudo-metric \( p_L \) immediately yields the following observations:

Remark 4. For every \( F \in \Delta^+ \) and every \( t > 0 \), the following hold (recall that \( u_0 = 1_{(0,\infty]} \in \Delta^+ \)):
\[
\begin{align*}
p_L(F, u_0) &= \inf\{h \in (0,1] : u_0(t) \leq F(t+h) + h, \ t \in J_h\} \\
&= \inf\{h \in (0,1] : F(h+) > 1 - k\}, \\
F(t) &> 1 - t \quad \text{iff} \quad p_L(F, u_0) < t.
\end{align*}
\]

Lemma 5. If \( F, G \in \Delta^+ \) and \( F \leq G \), then \( p_L(G, u_0) \leq p_L(F, u_0) \).

Proof. This is an immediate consequence of Remark 4.

Lemma 6. If \( \emptyset \neq A \subset \Delta^+ \), then \( G \in \Delta^+ \) where
\[
G(t) = \sup\{F(t) : F \in A\}.
\]

Proof. This follows from the information about lower semicontinuous functions.

Definition 7. Let \( q_L : \Delta^+ \times \Delta^+ \to I \) be given by the formula:
\[
q_L(F, G) = p_L(G, F) \quad \text{for all} \quad F, G \in \Delta^+.
\]
The function \( q_L \) is also a quasi-pseudo-metric on \( \Delta^+ \). The functions \( p_L \) and \( q_L \) are called conjugate and the structure on \( \Delta^+ \) generated by \( p_L \) is denoted by \( (\Delta^+, p_L, q_L) \).
Theorem 8. Given a structure \((\Delta^+, p_L, q_L)\), the function \(d_L : \Delta^+ \times \Delta^+ \to I\) defined by:
\[
d_L(F, G) = \max(p_L(F, G), q_L(F, G)) \quad \text{for } F, G \in \Delta^+.
\]
is a metric on the set \(\Delta^+\).

Proof. It suffices to show that the following condition holds:
\[
d_L(F, G) = 0 \iff F = G.
\]

Let \(t_0 \in (0, +\infty)\) and \(F(t_0) < G(t_0)\). Since \(F\) and \(G\) are left-continuous, there exists \(0 < t' < t_0\) such that \(F(t') < G(t')\). Now, take \(h < t_0 - t'\). By (1) and the fact that \(G\) is nondecreasing, we obtain the inequality:
\[
G(t') \leq G(t_0 - h) \leq F(t_0 - h + h) + h.
\]

If \(h \to 0\), then we get \(G(t_0 -) = G(t_0) \leq F(t_0)\), which is a contradiction. Taking into account that \(F(0) = G(0)\) and \(F(+\infty) = G(+\infty) = 1\), we eventually get the equality \(F(t) = G(t)\) for any \(t \in [0, +\infty]\). \(\square\)

Remark 9. Note that the metric given by Theorem 2 is equivalent to the metric defined by Schweizer and Sklar ([18], Definition 4.2.1).

Now, we state some facts related to the convergence in \((\Delta^+, d_L)\) and the weak convergence in the set \(\Delta^+\).

Definition 10. A sequence \(\{F_n\}\), where \(F_n \in \Delta^+\), is said to be weakly convergent to \(F \in \Delta^+\) (denoted by \(F_n \wto F\)) if and only if the sequence \(\{F_n(t)\}\) is convergent to \(F(t)\) for every point \(t\) of continuity of \(F\).

Let us recall the well-known fact that the convergence in every point of continuity of the function \(F\) fails to be equivalent to the convergence in any point of \((0, +\infty)\). Indeed, consider the sequence \(\{S(a-1/n, a)\}\), where \(a > 1\), and the function \(S(a-1/n, a)\) in \(\Delta^+\) is defined as follows:
\[
S(a-1/n, a)(t) = \begin{cases} 
0 & \text{if } 0 \leq t < a - \frac{1}{n}, \\
\frac{t-(a-\frac{1}{n})}{a-(a-\frac{1}{n})} & \text{if } t \in [a - \frac{1}{n}, a), \\
1 & \text{if } t \in [a, +\infty].
\end{cases}
\]

Notice that \(S(a-1/n, a) \wto u_a\), while, for every \(n \in \mathbb{N}\), we have
\[
S(a-1/n, a)(a) = 1 \neq 0 = u_a(a).
\]

Theorem 11. Let \(\{F_n\}_{n \in \mathbb{N}}\) be a sequence of the functions of \(\Delta^+\) and let \(F \in \Delta^+\). Then \(F_n \wto F\) if and only if \(d_L(F_n, F) \to 0\).
Proof. Assume that \( d_L(F_n, F) \to 0 \) and let \( t_0 > 0 \) be a point of continuity of \( F \). It follows that for sufficiently small \( h > 0 \), the interval \( (t_0 - h, t_0 + h) \) is contained in the interval \( (0, \frac{1}{n}) \) and the following hold:

\[
F(t_0) - h \leq F_n(t_0 + h) \quad \text{and} \quad F_n(t_0) \leq F(t_0 + h) + h
\]

for sufficiently large \( n \in \mathbb{N} \) and for \( t \in (0, \frac{1}{n}) \). Thus, by the monotonicity of \( F_n \) and \( F \) we obtain:

\[
F(t_0 - 2h) - f \leq F_n(t_0 - h) \leq F_n(t_0) \leq F_n(t_0 + h) \leq F(t_0 + 2h) + h.
\]

Since \( h \) is sufficiently small and \( F \) is continuous at \( t_0 \), it follows that \( F_n(t_0) \to F(t_0) \).

Conversely, assume that \( F_n \xrightarrow{w} F \). Let \( h \in (0, 1] \). Since the set of continuity points of \( F \) is dense in \([0, +\infty[\), there exists a finite set \( A = \{a_0, a_1, \ldots, a_p\} \) of continuity points of \( F \) such that: \( a_0 = 0, a_p \leq \frac{1}{n}, a_{m-1} < a_m \leq a_{m+1} + h \) for \( m = 1, 2, \ldots, p \). Since \( A \) is finite, for sufficiently large \( n \in \mathbb{N} \), we obtain \( |F_n(a_m) - F(a_m)| \leq h \) for all \( a_m \). Let \( t_0 \in (0, \frac{1}{n}) \). Then \( t_0 \in [a_{m-1}, a_m] \) for some \( m \). Therefore we have \( F(t_0) \leq F(a_m) \leq F_n(a_m) + h \leq F_n(t_0 + h) + h \), i.e. condition (13) is satisfied. By interchanging the role of \( F_n \) and \( F \) we obtain that \( F_n(t_0) \leq F(t_0 + h) + h \), which implies that \( d_L(F_n, F) \to 0 \). This completes the proof. \( \square \)

From the Helly’s theorem, it follows that, from every sequence in \( \Delta^+ \), one can select a subsequence which is weakly convergent. This fact and Theorem 11 yield the following result:

**Theorem 12.** The metric space \((\Delta^+, d_L)\) is compact, and hence complete.

### 3 \( t \)-Norms and Their Properties

Now, we shall give some definitions and properties of \( t \)-norms (Menger [11], [12], Schweizer, Sklar [18]) defined on the unit interval \( I = [0, 1] \). A \( t \)-norm \( T : I^2 \to I \) is an Abelian semigroup with unit, and the \( t \)-norm \( T \) is nondecreasing with respect to each variable.

**Definition 13.** Let \( T \) be a \( t \)-norm.

1. \( T \) is called a continuous \( t \)-norm if the function \( T \) is continuous with respect to the product topology on the set \( I \times I \).

2. The function \( T \) is said to be left-continuous if, for every \( x, y \in (0, 1] \), the following condition holds:

\[
T(x, y) = \sup\{T(u, v) : 0 < u < x, \ 0 < v < y\}.
\]
(3) The function $T$ is said to be right-continuous if, for every $x, y \in [0, 1)$, the following condition holds:

$$T(x, y) = \inf \{ T(u, v) : x < u < 1, \ y < v < 1 \}.$$ 

Note that the continuity of a $t$-norm $T$ implies both left and right-continuity of it.

**Definition 14.** Let $T$ be a $t$-norm. For each $n \in \mathbb{N}$ and $x \in I$, let 

$$x^0 = 1, \ x^1 = x \quad \text{and} \quad x^{n+1} = T(x^n, x), \quad \text{for all } n \geq 1.$$ 

Then the function $T$ is called an Archimedean $t$-norm if, for every $x, y \in (0, 1)$, there is an $n \in \mathbb{N}$ such that 

$$x^n < y, \quad \text{that is,} \quad x^n \leq y \quad \text{and} \quad x^n \neq y. \quad (2)$$

Note that $([0, 1], T)$ is a semigroup, we have 

$$T(x^n, x^m) = x^{n+m} \quad \text{for all } n, m \in \mathbb{N}.$$ 

From an immediate consequence of the above definition, we have the following:

**Lemma 15.** A continuous $t$-norms is Archimedean if and only if 

$$T(x, x) < x \quad \text{for all } x \in (0, 1).$$

**Proof.** Let $a \in (0, 1)$ be fixed and $y_n = a^n$. Since 

$$y_{n+1} = a^{n+1} = T(a^n, a) \leq T(a^n, 1) = a^n = y_n,$$

the sequence $\{y_n\}$ is non-increasing and bounded and so there exists $y = \lim_{n \to \infty} y_n$. Since $a^{2n} = T(a^n, a^n)$ and $T$ is continuous, we deduce that $y = T(y, y)$. 

If $T(x, x) < x$ for all $x \in (0, 1)$, then $y \in \{0, 1\}$ and, since $a^n \leq a < 1$, we have $y = 0$. 

Conversely, if there exists $a \in (0, 1)$ such that $T(a, a) = a$, then $a^{2n} = a$ for all $n \in \mathbb{N}$ and hence the sequence $\{a^n\}$ does not converge to 0. Therefore, $T(x, x) < x$ for all $x \in (0, 1)$. This completes the proof. 

**Lemma 16.** Let $T$ is a continuous $t$-norm and strictly increasing in $(0, 1)^2$ then it is Archimedean.

**Proof.** By the strict monotonicity of $T$, for any $x \in (0, 1)$, we have $T(x, x) < x$. 

**Definition 17.** Let $T$ be a $t$-norm. Then $T$ is said to be positive if $T(x, y) > 0$ for all $x, y \in (0, 1]$. 

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Note that every \( t \)-norm satisfying the assumption of Lemma 16 is positive.

We shall now establish the notation related to a few most important \( t \)-norms defined by:

\[
M(x, y) = \min(x, y) = x \land y
\]  

(3) for all \( x, y \in I \). The function \( M \) is continuous and positive, but is not Archimedean (in fact, it fails to satisfy the strict monotonicity condition).

\[
\Pi(x, y) = x \cdot y
\]  

(4) for all \( x, y \in I \). The function \( \Pi \) is strictly increasing and continuous and hence it is a positive archimedean \( t \)-norm.

\[
W(x, y) = \max(x + y - 1, 0)
\]  

(5) for all \( x, y \in I \). The function \( W \) is continuous and Archimedean, but it is not positive and hence it fails to be a strictly increasing \( t \)-norm.

\[
Z(x, y) = \begin{cases} 
x & \text{if } x \in I \text{ and } y = 1, 
y & \text{if } x = 1 \text{ and } y \in I, 
0 & \text{if } x, y \in [0, 1). 
\end{cases}
\]  

(6)

The function \( Z \) is Archimedean and right-continuous, but it fails to be left-continuous.

For any \( t \)-norm \( T \), we have

\[ Z \leq T \leq M \]  

in particular

\[ Z < W < \Pi < M. \]

4 Triangle Functions and Their Properties

In this section, we shall now present some properties of the triangle functions on \( \Delta^+ \) (Šerstnev [19], Schweizer, Sklar [18]).

The ordered pair \((\Delta^+, \ast)\) is an Abelian semigroup with the unit \( u_0 \in \Delta^+ \) and the operation \( \ast : \Delta^+ \times \Delta^+ \rightarrow \Delta^+ \) is a nondecreasing function. We note that \( u_\infty \in \Delta^+ \) is a zero of \( \Delta^+ \). Indeed, we obtain

\[ u_\infty \leq u_\infty \ast F \leq u_\infty \ast u_0 = u_\infty \text{ for all } F \in \Delta^+. \]

**Definition 18.** Let \( T(\Delta^+, \ast) \) denote the family of all triangle functions on the set \( \Delta^+ \). Then the relation \( \leq \) defined by

\[ \ast_1 \leq \ast_2 \iff F \ast_1 G \leq F \ast_2 G \text{ for all } F, G \in \Delta^+ \text{ partially orders the family } T(\Delta^+, \ast). \]  

(7)
Now, we are going to define the next relation in the $T(\Delta^+, \ast)$. It will be denoted by $\gg$ and is defined as follows:

\[ \ast_1 \gg \ast_2 \iff \forall F, G, P, Q \in \Delta^+ \quad [(F \ast_2 P) \ast_1 (G \ast_2 R)] \geq [(F \ast G) \ast_2 (P \ast R)]. \quad (8) \]

By putting $G = P = u_0$ we obtain $F \ast_1 R \geq F \ast_2 R$ for $F, R \in \Delta^+$ and hence $\ast_1 \geq \ast_2$.

**Theorem 19.** Let $T$ be a left-continuous $t$-norm. Then the function $T : \Delta^+ \times \Delta^+ \to \Delta^+$ defined by

\[ T(F, G)(t) = T(F(t), G(t)) \quad (9) \]

for any $t \in [0, +\infty]$ is a triangle function on the set $\Delta^+$.

**Theorem 20.** For every triangle function $\ast$, the following inequality holds:

\[ \ast \leq M, \]

where $M$ is the $t$-norm of Definition 17.

**Proof.** For every $F, G \in \Delta^+$, we have by definition of $(\Delta^+, \ast)$, $F \ast G \leq F \ast u_0 = F$ and, by symmetry, also $F \ast G \leq G$. Thus, for every $t \in [0, +\infty]$, we have

\[ (F \ast G)(t) \leq M(F(t), G(t)) = M(F, G)(t). \quad (10) \]

**Theorem 21.** If $T$ is a left-continuous $t$-norm, then the function $\ast_T : \Delta^+ \times \Delta^+ \to \Delta^+$ defined by

\[ F \ast_T G(t) = \sup \{T(F(u), G(s)) : u + s = t, \ u, s > 0\} \quad (11) \]

is a triangle function on $\Delta^+$.

**Proof.** The function $F \ast_T G \in \Delta^+$ is nondecreasing and satisfies the condition $F \ast_T G(+\infty) = 1$ for all $F, G \in \Delta^+$. Thus it suffices to check that $F \ast_T G$ is left-continuous, i.e., for every $t \in (0, +\infty)$ and $h > 0$, there exists $0 < t_1 < t$ such that

\[ F \ast_T G(t_1) > F \ast_T G(t) - h. \]

Let $t \in (0, +\infty)$. Then there exist $u, s > 0$ such that $u + s = t$ and

\[ T(F(u), G(s)) > F \ast_T G(t) - \frac{h}{2}. \quad (12) \]

By the left-continuity of $F, G$ and the $t$-norm $T$, it follows that there are numbers $0 \leq u_1 < u$ and $0 \leq s_1 \leq s$ such that

\[ T(F(u_1), G(s_1)) > T(F(u), G(s)) - \frac{h}{2}. \quad (13) \]
Now, put $t_1 = u_1 + s_1$. Then $t_1 < t$ and, by (11), we obtain

$$F * T G(t) \geq T(F(u_1), G(s_1)). \quad (14)$$

This completes the proof.

**Theorem 22.** Let $T$ be a continuous $t$-norm. Then the triangular functions $* T$ and $T$ are uniformly continuous on $(\Delta^+, d_L)$.

**Proof.** (see Theorem 7.2.8 [18]) Let us observe that the continuity of the $t$-norm $T$ implies its uniform continuity on $I \times I$ with the product topology. Take an $h \in (0, 1)$. Then there exists $s > 0$ such that

$$T(\text{Min}(z + s, 1), w) < T(z, w) + \frac{h}{4}$$

and

$$T(z, \text{Min}(w + s, 1)) < T(z, w) + \frac{h}{4} \quad (15)$$

for all $z, w \in I$. Let $u < 1/s$ and $v < 1/s$ be such that $u + v < 2/h$. Next, by (11), for every $F, G \in \Delta^+$ and $t \in (0, 2/h)$, there exist $u, v > 0$ such that $u + v = t$ and

$$F * T G(t) < T(F(u), G(v)) + \frac{h}{4}.$$

Now, let $F_1 \in \Delta^+$ be such that $d_L(F, F_1) < s$, which means that

$$F(u) \leq F_1(u + s) + s$$

for all $u \in (0, \frac{1}{s})$. Since $u + v = t < 2/h$, we have $u < 2/h$. Therefore, we obtain

$$F * T G(t) < T(\text{Min}(F_1(u + s) + s, 1), G(v)) + \frac{h}{2}$$

$$< T(F_1(u + s), G(v)) + \frac{h}{2}$$

and

$$F * T G(t) < F_1 * T G(u + s + v) + \frac{h}{2}$$

$$\leq F_1 * T G(u + v + \frac{h}{2}) + \frac{h}{2}$$

$$= F_1 * T G(t + \frac{h}{2}) + \frac{h}{2}.$$  

Thus, by (1), we have

$$p_L(F_1 * T G, G) \leq \frac{h}{2}, \quad q_L(F * T G, F_1 * T G) \leq \frac{h}{2}.$$  

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and so we have
\[ d_L(F_1 * T G, F * T G) \leq \frac{h}{2}. \]

If \( d_L(G, G_1) < s \), then we have
\[ d_L(F_1 * T G_1, F * T G) \leq \frac{h}{2} \]
and so let \( F, F_1, G, G_1 \in \Delta^+ \) satisfy the conditions \( d_L(F, F_1) < s \) and \( d_L(G, G_1) < s \). Then we have
\[
\begin{align*}
&d_L(F_1 * T G_1, F * T G) \\
&\leq d_L(F_1 * T G_1, F_1 * T G) + d_L(F_1 * T G, F * T G) \\
&\leq \frac{h}{2} + \frac{h}{2} = h.
\end{align*}
\]

It follows that the triangle function \( *_T \) is uniformly continuous in the space \( (\Delta^+, d_L) \). The second part is a simple restatement of the first one. This completes the proof.

**Remark 23.** There exist triangle functions which are not continuous on \( (\Delta^+, d_L) \). Among them, there is the function \( *_Z \) of (11) and (6). Indeed, this can be seen by the following example.

Let \( F_n(t) = 1 - e^{-\frac{t}{n}} \), where \( n \in \mathbb{N} \). Then
\[ F_n \overset{w}{\to} u_0 \]
while the sequence \( \{F_n * Z F_n\} \) fails to be weakly convergent to \( u_0 * Z u_0 \) because \( F_n * Z F_n = u_\infty \) for all \( n \in \mathbb{N} \). We note that this example actually shows much more: the triangle function \( *_Z \) is not continuous on \( (\Delta^+, d_L) \). In particular, it is not continuous at the point \( (u_0, u_0) \).

We finish this section by showing a few properties of the relation defined in (8) in the context of triangle functions (22).

**Lemma 24.** If \( T_1 \) and \( T_2 \) are continuous \( t \)-norms, then triangle functions \( T_1, T_2 \) given by (9),
\[ T_1 \gg T_2 \text{ if and only if } *_{T_1} \gg *_{T_2}. \]

**Lemma 25.** If \( T \) is a continuous \( t \)-norm and \( *_T \) is the triangle function of (9), then
\[
\begin{align*}
T &\gg *_T, & (16) \\
M &\gg * & \text{for all triangle functions } *.
\end{align*}
\]
5 Properties of \( P_{qpM} \)-Spaces

First, we give the definition of \( P_{qpM} \)-spaces and some properties of \( P_{qpM} \)-spaces and others.

**Definition 26.** ([2, 3, 4, 6]) By a \( P_{qpM} \)-space we mean an ordered triple \((X, P, \star)\), where \( X \) is a nonempty set, the operation \( \star \) is triangle function and \( P : X^2 \to \Delta^+ \) satisfies the following conditions (by \( P_{xy} \) we denote the value of \( P \) at \( (x, y) \in X^2 \)): for all \( x, y, z \in X \),

\[
\begin{align*}
P_{xx} &= u_0, \quad (18) \\
P_{xy} \star P_{yz} &\leq P_{xz}. \quad (19)
\end{align*}
\]

If \( P \) satisfies also the additional condition:

\[
P_{xy} \neq u_0 \quad \text{if} \quad x \neq y,
\]

then \((X, P, \star)\) is called a probabilistic quasi-metric space (denoted by \( PqM \)-space).

Moreover, if \( P \) satisfies the condition of symmetry:

\[
P_{xy} = P_{yx},
\]

then \((X, P, \star)\) is called a probabilistic metric space (denoted by \( PM \)-space).

**Definition 27.** [6] Let \((X, P, \star)\) be a \( P_{qpM} \)-space and let \( Q : X^2 \to \Delta^+ \) be defined by the following condition:

\[
Q_{xy} = P_{yx}
\]

for all \( x, y \in X \). Then the ordered triple \((X, Q, \star)\) is also a \( P_{qpM} \)-space. We say that the function \( P \) is a conjugate \( P_{qp} \)-metric of the function \( Q \). By \((X, P, Q, \star)\) we denote the structure generated by the \( P_{qp} \)-metric \( P \) on \( X \).

Now, we shall characterize the relationships between \( P_{qp} \)-metrics and probabilistic pseudo-metrics.

**Lemma 28.** Let \((X, P, Q, \star)\) be a structure defined by a \( P_{qp} \)-metric \( P \) and let \( \star_1 \gg \star \)

\[
\text{Then the ordered triple } (X, F^{\star_1}, \star) \text{ is a probabilistic pseudo-metric space (denoted by } \text{PPM-space)}\) whenever the function \( F^{\star_1} : X^2 \to \Delta^+ \) is defined in the following way:

\[
F^{\star_1}_{xy} = P_{xy} \star_1 Q_{xy}
\]

for all \( x, y \in X \). If, additionally, \( P \) satisfies the condition:

\[
P_{xy} \neq u_0 \quad \text{or} \quad Q_{xy} \neq u_0 \quad (24)
\]

for \( x \neq y \), then \((X, F^{\star_1}, \star)\) is a \( PM \)-space.

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Proof. For any $x, y \in X$, we have

$$F_{xy}^1 \in \Delta^+ \quad \text{and} \quad F_{xy}^1 = F_{yx}^1.$$  

By (18), we obtain

$$F_{xy}^1 = P_{xx} *_1 Q_{xx} = u_0 *_1 u_0 = u_0.$$  

Next, by (19) and (22) and the monotonicity of triangle function, we obtain

$$F_{xy}^1 = P_{xy} *_1 Q_{xy} \geq (P_{xz} * P_{zz}) *_1 (Q_{xz} * Q_{zy}) \geq (P_{xz} *_1 Q_{zz}) * (P_{zy} *_1 Q_{zy}) = F_{xz}^1 * F_{zy}^1.$$  

The proof of the second part of the theorem is a direct consequence of the fact that the conditions (24) and (23) both imply the statement that

$$F_{xy}^1 = P_{xy} *_1 Q_{xy} = u_0 \quad \text{if and only if} \quad P_{xy} = Q_{xy} = u_0.$$  

It follows that, whenever $x \neq y$, $P_{xy} \neq u_0$ or $Q_{xy} \neq u_0$ and hence $P_{xy} *_1 Q_{xy} \neq u_0$. This completes the proof.

**Remark 29.** For an arbitrary triangle function (22), we know, by Lemma 25, that $M \gg *$. Using (23), we have

$$F_{P\vee Q} = F^M(x, y) \geq F^*_1(x, y) \quad \text{for all} \quad x, y \in X.$$  

for all $x, y \in X$.

The function $F^M$ will be called the natural probabilistic pseudo-metric generated by the $P_{qp}$-metric $P$. It is the ”greatest” among all the probabilistic pseudo-metrics generated by $P$.

**Definition 30.** Let $X$ be a nonempty set and $P : X^2 \to D^+$, where $D^+ = \{F \in \Delta^+ ; \lim_{t \to \infty} F(t) = 1\}$ and $T$ is $t$-norm. The triple $(X, P, T)$ is called a quasi-pseudo-Menger space if it satisfies the following axioms:

$$P_{xx} = u_0 \quad \text{(26)}$$

$$P_{xy}(u + v) \geq T(P_{xz}(u), P_{zy}(v)) \quad \text{for all} \quad x, y, z \in X \quad \text{and} \quad u, v \in R. \quad \text{(27)}$$

If $P$ satisfies also the additional condition:

$$P_{xy} \neq u_0 \text{ if } x \neq y$$  

then $(X, P, T)$ is a quasi-Menger space.

Moreover, if $P$ satisfies the condition of symmetry $P_{xy} = P_{yx}$, then $(X, P, T)$ is called a Menger-space (see [11, 12]).
Definition 31. Let \((X, p)\) be a quasi-pseudo-metric-space and \(G \in D^+\) be distinct from \(u_0\). Define a function \(G_p : X^2 \to D^+\) by
\[
G_p(x, y) = G\left(\frac{t}{p(x, y)}\right) \quad \text{for all } t \in R^+
\]
and \(G\left(\frac{1}{2}\right) = G(\infty) = 1\), for \(t > 0\), \(G(0) = G(\frac{1}{2}) = G(0) = 0\). Then \((X, G_p)\) is called a \(P\)-simple space generated by \((X, p)\) and \(G\).

Theorem 32. Every \(P\)-simple space \((X, G_p)\) is a quasi-pseudo-Menger space respect to the \(t\)-norm \(M\).

Proof. For all \(x, y, z \in X\), by the triangle condition for the quasi-pseudo-metric \(p\), we have
\[
p(x, y) \geq p(x, y) + p(y, z).
\]
Assume, that all at \(p(x, z), p(x, y)\) and \(p(y, z)\) are distinct from zero. For any \(t_1, t_2 > 0\), we obtain
\[
\frac{t_1 + t_2}{p(x, z)} \geq \frac{t_1 + t_2}{p(x, y) + p(y, z)}
\]
and hence we infer that
\[
\max\left\{\frac{t_1}{p(x, y)}, \frac{t_2}{p(y, z)}\right\} \geq \frac{t_1 + t_2}{p(x, y) + p(y, z)} \geq \min\left\{\frac{t_1}{p(x, y)}, \frac{t_2}{p(y, z)}\right\}.
\]
This inequality and the monotonicity of \(G\) imply that
\[
G_p(x, z)(t_1 + t_2) \geq \min(G_p(x, y)(t_1), G_p(y, z)(t_2)),
\]
for \(t_1, t_2 \geq 0\). This completes the proof. \(\square\)

6 The family of \(Pqp\)-metrics on a set \(X\)

Definition 33. Let \(P[X, \ast]\) denote the family of all \(Pqp\)-metrics defined on a set \(X\) with respect to a triangle function \(\ast\). Define on \(X\) a relation \(\prec\) in the following way:
\[
P_1 \prec P_2 \quad \text{iff} \quad P_1(x, y) \geq P_2(x, y) \quad \text{for all } x, y \in X.
\]

We note that \(\prec\) is a partial order on the family \(P[X, \ast]\). We distinguish elements \(P_0\) and \(P_\infty\) in it:
\[
P_0(x, y) = u_0 \quad \text{for all } x, y \in X,
\]
\[
P_\infty(x, y) = u_0 \quad \text{and} \quad p_\infty(x, y) = u_\infty \quad \text{for } x \neq y.
\]
We note that \(P_0 \prec P \prec P_\infty\) for every \(P \in P[X, \ast]\).
Now, we give the definition of certain binary operation $\oplus$ on $P[X, \ast]$. Let for all $P_1, P_2 \in P[X, \ast]$:

$$P_1 \oplus P_2(x, y) = P_1(x, y) \ast P_2(x, y), \quad x, y \in X. \quad (35)$$

We note that $P_1 \oplus P_2 \in P[X, \ast]$. Indeed, we prove the condition (18) directly:

$P_1 \oplus P_2(x, x) = P_1(x, x) \ast P_2(x, x) = u_0$.

The condition (19) follows from $F \ast u_0 = F$ when applied to $P_1$ and $P_2$:

$$P_1 \oplus P_2(x, y) = P_1(x, y) \oplus P_2(x, y)$$

$$\geq (P_1(x, y) \ast P_1(z, y)) \ast (P_2(z, z) \ast P_2(z, z))$$

$$= (P_1(x, z) \ast P_2(x, z)) \ast (P_1(z, y) \ast P_2(z, y))$$

$$= (P_1 \oplus P_2(x, y)) \ast (P_1(z, y) \oplus P_2(z, y)).$$

This shows that $P_1 \oplus P_2$ is a $Pqp$-metric. Notice also that for each $P \in P[X, \ast]$ the following property holds:

$$P_0 \oplus P = P. \quad (36)$$

Indeed, $P_0 \oplus P(x, y) = u_0 \ast P_{xy} = P(x, y)$.

The operation $\oplus$ is also commutative and associative. This is a consequence of the form of (22). Thus we have the following corollary:

**Lemma 34.** The ordered triple $(P[X, \ast], \oplus, P_0)$ is an Abelian semi-group with respect to the operation $\ast$, and has the neutral element $P_0$.

The following gives a relationship between the relation $\prec$ and the operation $\oplus$.

**Lemma 35.** Let $(P[X, \ast], \oplus, P_0)$ be as in Lemma 35. Then, for all $P, P_1, P_2 \in P[X, \ast]$, the following hold:

$$P_0 \prec P, \quad (37)$$

$$P_1 \oplus P \prec P_2 \oplus P \quad \text{whenever} \quad P_1 \prec P_2. \quad (38)$$

**Proof.** That the first property holds true follows from the Definition 33. The relation $P_1 \prec P_2$ means, by (32), that $P_1(x, y) \geq P_2(x, y), \quad x, y \in X$. Since 22 is a monotone function, we get $P_1(x, y) \ast P(x, y) \geq P_2(x, y) \ast P(x, y)$. This shows the validity of the second condition.

Let us define in $P[X, \ast]$ get another operation, denoted by $\vee$. For any $P_1, P_2 \in P[X, \ast]$, let

$$P_1 \vee P_2 = \min(P_1, P_2) = M(P_1, P_2). \quad (39)$$

By Lemma 5 it follows that $M \gg \ast$ for all $\ast$. Thus we have $P_1 \vee P_2 \in P[X, \ast]$.

The following accounts for some properties of the operation $\vee$. 

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Lemma 36. The ordered pair \((P[X, *], \vee)\) is a \(\vee\)-semi-lattice (see Grätzer [4]) satisfying the following conditions: for all \(P, P_1, P_2 \in P[X, *]\),

\[
P_1 \prec P_2 \iff P_1 \vee P_2 = P_2,
\]

\[(P \oplus P_1) \vee (P \oplus P_2) \prec P \oplus (P_1 \vee P_2).\]

Proof. \(P \vee P = M(P, P) = P\), hence \(\vee\) satisfies the indempotency. It is also commutative. This yields the first part of the Lemma. Next, observe that if \(P_1 \prec P_2\), then \(P_1(x, y) \geq P_2(x, y), x, y \in X\). Thus \(M(P_1, P_2) = P_2\). We have shown the first property. For a proof of the second one notice that \(P_1 \prec P_1 \vee P_2\) and \(P_2 \prec P_1 \vee P_2\).

By (38) we get \(P \oplus P_1 \prec P \oplus (P_1 \vee P_2)\) and \(P \oplus P_1 \prec P \oplus (P_1 \vee P_2)\). Since \((P[X, *], \vee)\) is a \(\vee\)-semilattice, the condition (41) follows. This completes the proof.

7 Families of quasi-pseudo-metrics generated by \(PqpM\)-metrics

We shall now give some classification of \(PqpM\)-spaces with respect to the so-called "triangle condition".

Definition 37. Let \(X\) be a nonempty set. Let \(P : X^2 \to \Delta^+\) satisfy the condition (18) and let, for all \(x, y, z \in X\), the following implication hold:

\[
\text{If } P_{xy}(t_2) = 1 \text{ and } P_{yz}(t_2) = 1, \text{ then } P_{xy}(t_1 + t_2) = 1 \text{ for all } t_1, t_2 > 0.
\]

Then the ordered pair \((X, P)\) is called a statistical quasi-pseudo-metric space. We write \(SpqM\)-space.

Topics related to the "triangle condition" belong to the most important ones in the theory of PM-spaces. We mention here the most important papers in a chronological order (see Menger [11], Wald [21], Schweizer and Sklar [16, 17], Muşări and Serstnev [13], Brown [5], Istrăţescu [8], Radu [15]).

Definition 38. Let \(T\) be \(t\)-norm ones a function \(P : X^2 \to \Delta^+\) is assumed to satisfy the condition (18) and, for all \(x, y, z \in X\), let

\[
P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)), \quad t_1, t_2 > 0.
\]

Then \((X, P, T)\) is called a quasi-pseudo-Menger space.

Condition (44) is called a Menger condition and comes from a paper by Schweizer and Sklar ([13, 14]). It is modification of an inequality of Menger ([7, 8]).

Lemma 39. Each quasi-pseudo-Menger space is an \(SpqM\)-space.
Proof. Assume \( P_{xy}(t_1) = 1 \) and \( P_{yz}(t_2) = 1 \) for any \( t_1, t_2 > 0 \). By (M.2), we have
\[
P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) = T(1, 1) = 1.
\]
Let \( X \) be a nonempty set and let \( P : X^2 \to \Delta^+ \) satisfy the condition (18). For each \( a \in [0, 1) \) define \( p_a : X \to \mathbb{R} \) by
\[
p_a(x, y) = \inf \{ t > 0 : P_{xy}(t) > a \text{ for } x, y \in X \}.
\]
Since \( P_{xy} \) is nondecreasing and let-continuous, the following equivalence holds for \( x, y \in X \) and \( a \in [0, 1) \):
\[
p_a(x, y) < t \iff P_{xy}(t) > a.
\]
The family \( D(X, P, a) \) of all functions \( p_a \) has the following properties which are the consequences of (46):
\[
p_a(x, y) \geq 0, \quad p_a(x, x) = 0 \text{ for } x, y \in X \text{ and } a \in [0, 1). \tag{47}
\]
Under the additional assumption that \( P \) satisfies the following condition: for all \( a \in [0, 1) \),
\[
P_{xy}(t_1) > a \text{ and } P_{yz}(t_2) > a \Rightarrow P_{xz}(t_1 + t_2) > a \tag{49}
\]
for all \( x, y, z \in X \) and \( t_1, t_2 > 0 \),
then for every \( a \in [0, 1) \) the function \( p_a \) satisfies
\[
p_a(x, z) \leq p_a(x, y) + p_a(y, z) \text{ for } x, y, z \in X. \tag{51}
\]
This completes the proof.

As a consequence of this fact we conclude the following:

**Lemma 40.** The family \( D(X, P, a) \) of all the functions \( p_a \) with \( a \in [0, 1) \) is a family of quasi-pseudo-metrics if and only if the function \( P \) satisfies (5.3.5). For any \( a \in (0, 1) \), \( p_a \) is a quasi-metric if and only if \( p_{xy}(0+) < a \) for all \( x \neq y \) in \( X \).

Proof. For the first assertion, it suffices to show the triangle condition (51). Given an arbitrary \( s > 0 \), put \( t_1 = p_a(x, y) + \frac{s}{2} \) and \( t_2 = p_a(y, z) + \frac{s}{2} \). By (46) we then have \( P_{xy}(t_1) > a \) and \( P_{yz}(t_2) > a \). By (49) this yields the inequality \( P_{xz}(t_1 + t_2) > a \) which is equivalent to \( p_a(x, z) < t_1 + t_2 = p_a(x, y) + p_a(y, z) + s \). Since \( s \) is arbitrary, we obtain the required inequality (51).

The second assertion follows from the fact that \( p_a(x, y) = 0 \) if and only if \( P_{xy}(t) > a \) for all \( t > 0 \), i.e., when \( P_{xy}(0+) \geq a \). The proof is complete.
Remark 41. Observe that if $P : X^2 \to \Delta^+$ satisfies the conditions (18) and (49), then $(X, P)$ is a statistical quasi-pseudo-metric space.

Indeed, let $P_{xy}(t_1) = 1$ and $P_{yz}(t_2) = 1$. Then it follows by (49) that $P_{xz}(t_1 + t_2) > a$ for all $a \in [0, 1)$. Thus $P_{xz}(t_1 + t_2) = 1$. Thus $P_{xz}(t_1 + t_2) = 1$. This shows that the condition (37) of Definition 37 holds true.

The following observation is a consequence of the preceding remark:

Corollary 42. Let the function $P$ satisfy the conditions (18) and (49) and let, for every $x, y \in X$, there exists a number $t_{xy} < \infty$ such that $P_{xy}(t_{xy}) = 1$. Then the function $p_a$ is a quasi-pseudo-metric for every $a \in [0, 1]$. In particular, $p_1 : X^2 \to R$ is given by the following formula:

$$p_1(x, y) = \inf \{ t > 0 : P_{xy}(t) = 1 \text{ for } x, y \in X \}. \quad (52)$$

Proof. Let $s > 0$. Let $t_1 = p_1(x, y) + \frac{s}{2}$ and $t_2 = p_1(y, z) + \frac{s}{2}$. Then $P_{xy}(t_1) = 1$ and $P_{yz}(t_2) = 1$, and thus, by (45), we have $P_{xz}(t_1 + t_2) = 1$. We now have $p_1(x, z) < t_1 + t_2 = p_1(x, y) + p_1(y, z) + s$. Finally, the condition (51) is satisfied on account of $s$ being arbitrary.

Remark 43. Let $(X, P, \ast_M)$ be a quasi-pseudo-Menger space. Then the function $P$ satisfies the condition (49). Indeed, let $P_{xy}(t_1) > a$ and $P_{yz}(t_2) > a$. By (M.2), we get $P_{xz}(t_1 + t_2) \geq \min(P_{xy}(t_1), P_{yz}(t_2)) > \min(a, a) = a$.

The following is an immediate consequence of Lemma 40 and Remark 43:

Corollary 44. If $(X, P, \ast_M)$ is a quasi-pseudo-Menger space, then the family $D(X, P, a)$ defined in (45) is a family of the quasi-pseudo-metrics on $X$ for all $a \in [0, 1)$.

Theorem 45. Let $(X, P, T)$ be a quasi-pseudo-Menger space. Let the function $d(x) = T(x, x)$ be strictly increasing and continuous on some interval $[a, b] \subset I$. Then, if $T(a, a) = a$, then the function $p_a$ of (45) is a quasi-pseudo-metric in $X$. For $a > 0$, $p_a$ is a quasi-metric in $X$ if and only if $P_{xy}(0+) < a$ whenever $x \neq y$.

Proof. It suffices to show that the property (49) holds true for any $a \in [0, 1)$, which satisfies the assumption of the theorem.

Let $P_{xy}(t_1) > a$ and $P_{yz}(t_2) > a$. Since $P_{xy}$ and $P_{yz}$ are nondecreasing and left-continuous, there exists $s > 0$ such that $a + s < b$, $P_{xy}(t_1) > a + s$ and $P_{yz}(t_2) > a + s$.

The properties of the function $d(x) = T(x, x)$ and the condition (44) yield the inequality $P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) \geq T(a + s, a + s) > a$. The assertion is now a consequence of Lemma 40.

Theorem 46. Let $(X, P, T)$ be a quasi-pseudo-Menger space such that $T \geq I$. Then the family $D(X, P, p_a)$ of all the functions $p_a : X^2 \to R$ given by

$$p_a(x, y) = \inf \{ t > 0 : P_{xy}(t) > a(t), \quad x, y \in X \}, \quad (53)$$

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consists of quasi-pseudo-metrics, if all the functions \( a : [0, +\infty] \to [0, 1] \) are defined by the following formula:

\[
a(t) = \begin{cases} 
  e^{-at}, & t \in [0, +\infty), \\
  0, & t = +\infty, \text{ where } a \in (0, +\infty). 
\end{cases}
\]  

(54)

The functions \( p_a \) are quasi-metrics if and only if \( P_{xy}(0+) < 1 \) whenever \( x \neq y \).

**Proof.** Observe that for every \( a \in (0, +\infty) \) the functions are strictly decreasing. Let \( t_1 = p_a(x, y) + \frac{s}{2} \) and \( t_2 = p_a(y, z) + \frac{s}{2} \), \( s > 0 \). This means that by (46) the following inequalities hold:

\[
\begin{align*}
P_{xy}(t_1) &\geq a(p_a(x, y)) > a(t_1), \\
P_{yz}(t_2) &\geq a(p_a(y, z)) > a(t_2).
\end{align*}
\]

By (44) and the inequality \( T \gg \Pi \), we obtain

\[
P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) \\
\geq T(a(p_a(x, y), a(p_a(y, z)))) \\
\geq \Pi(a(p_a(x, y), a(p_a(y, z)))) \\
> \Pi(a(t_1), a(t_2)) = e^{-a t_1} \cdot e^{-a t_2} \\
= e^{-a(t_1 + t_2)} = a(t_1 + t_2).
\]

This means that \( p_a(x, z) < t_1 + t_2 = p_a(x, y) + p_a(y, z) + s \) for any \( s > 0 \), so that the triangle condition holds. This completes the proof. \( \square \)

**Theorem 47.** Let \( (X, P, T) \) be a quasi-pseudo-Menger space with \( T \geq W \) (28). Then the family \( D(X, P, p_a) \) of all the functions \( p_a \) of (53) consists of quasi-pseudo-metrics, provided the functions \( a : [0, +\infty] \to [0, 1] \) are defined by the following formula:

\[
a(t) = \begin{cases} 
  1 - \frac{t}{a}, & t \in [0, a], \\
  0, & t > a \text{ where } a \in (0, +\infty). 
\end{cases}
\]  

(55)

**Proof.** Let \( t_1 = p_a(x, y) + \frac{s}{2} \) and \( t_2 = p_a(y, z) + \frac{s}{2} \), \( s > 0 \). By (46), we have

\[
P_{xy}(t_1) \geq a(p_a(x, y)) > a(t_1) \quad \text{and} \quad P_{yz}(t_2) \geq a(p_a(y, z)) > a(t_2).
\]

By (44) and the inequality \( T \geq W \), we get

\[
P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) \geq T(a(p_a(x, y), a(p_a(y, z)))) \\
\geq W(a(p_a(x, y), a(p_a(y, z)))) > W(a(t_1), a(t_2)) \\
= \max \left( 1 - \frac{t_1}{a} + 1 - \frac{t_2}{a} - 1, 0 \right) \\
= 1 - \frac{t_1 + t_2}{a} = a(t_1 + t_2).
\]
Therefore $p_a(x, z) < t_1 + t_2 = p_a(x, y) + p_a(y, z) + s$ for every $s > 0$, i.e., the triangle inequality holds.

Acknowledgement. The authors would like to thank the referees and the area editor Prof. Barnabas Bede for giving useful comments and suggestions for improving the paper.

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