The Relation Between the Associate Almost Complex Structure to $HM'$ and $(HM', S, T)$-Cartan Connections

Ebrahim ESRAFILIAN and Hamid Reza SALIMI MOGHADDAM

Department of Pure Mathematics, Faculty of Mathematics, Iran University of Science and Technology, Narmak-16, Tehran, Iran
E-mail: salimi_m@iust.ac.ir

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Abstract. In the present paper, the $(HM', S, T)$-Cartan connections on pseudo-Finsler manifolds, introduced by A. Bejancu and H.R. Farran, are obtained by the natural almost complex structure arising from the nonlinear connection $HM'$. We prove that the natural almost complex linear connection associated to a $(HM', S, T)$-Cartan connection is a metric linear connection with respect to the Sasaki metric $G$. Finally we give some conditions for $(M', J, G)$ to be a Kähler manifold.

Key words: almost complex structure; Kähler and pseudo-Finsler manifolds; $(HM', S, T)$-Cartan connection

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1 Introduction

Almost complex structures are important structures in differential geometry [8, 9, 11]. These structures have found many applications in physics. H.E. Brandt has shown that the spacetime tangent bundle, in the case of Finsler spacetime manifold, is almost complex [4, 5, 6]. Also he demonstrated that in this case the spacetime tangent bundle is complex provided that the gauge curvature field vanishes [3]. In [1, 2], for a pseudo-Finsler manifold $F^m = (M, M', F^*)$ with a nonlinear connection $HM'$ and any two skew-symmetric Finsler tensor fields of type (1, 2) on $F^m$, A. Bejancu and H.R. Farran introduced a notion of Finsler connections which named “$(HM', S, T)$-Cartan connections”. If, in particular, $HM'$ is the canonical nonlinear connection $GM'$ of $F^m$ and $S = T = 0$, the Finsler connection is called the Cartan connection and it is denoted by $FC^* = (GM', \nabla^*)$ (see [1]). They showed that $\nabla^*$ is the projection of the Levi-Civita connection of the Sasaki metric $G$ on the vertical vector bundle. Also they proved that the associate linear connection $D^*$ to the Cartan connection $FC^*$ is a metric linear connection with respect to $G$ [1]. In this paper we obtain the $(HM', S, T)$-Cartan connections by using the natural almost complex structure arising from the nonlinear connection $HM'$, then the natural almost complex linear connection associated to a $(HM', S, T)$-Cartan connection is defined. We prove that the natural almost complex linear connection associated to a $(HM', S, T)$-Cartan connection is a metric linear connection with respect to the Sasaki metric $G$. Kähler and para-Kähler structures associated with Finsler spaces and their relations with flag curvature were studied by M. Crampin and B.Y. Wu (see [7, 12]). They have found some interesting results on this matter. In [12], B.Y. Wu gives some equivalent statements to the Kählerity of $(M', G, J)$. In the present paper we give other conditions for the Kählerity of $(M', G, J)$, which extend the previous results.
2 The associate almost complex structure to $HM'$

Let $M$ be a real $m$-dimensional smooth manifold and $TM$ be the tangent bundle of $M$. Let $M'$ be a nonempty open submanifold of $TM$ such that $\pi(M') = M$ and $\theta(M) \cap M' = \emptyset$, where $\theta$ is the zero section of $TM$. Suppose that $F^m = (M, M', F^*)$ is a pseudo-Finsler manifold where $F^* : M' \rightarrow \mathbb{R}$ is a smooth function which in any coordinate system $\{(U', \Phi') : x', y'\}$ in $M'$, the following conditions are fulfilled:

- $F^*$ is positively homogeneous of degree two with respect to $(y^1, \ldots, y^m)$, i.e., we have
  $$F^*(x^1, \ldots, x^m, ky^1, \ldots, ky^m) = k^2F^*(x^1, \ldots, x^m, y^1, \ldots, y^m)$$
  for any point $(x, y) \in (\Phi', U')$ and $k > 0$.

- At any point $(x, y) \in (\Phi', U')$, $g_{ij}$ are the components of a quadratic form on $\mathbb{R}^m$ with $q$ negative eigenvalues and $m - q$ positive eigenvalues, $0 < q < m$ (see [1]).

Consider the tangent mapping $\pi_* : TM' \rightarrow TM$ of the submersion $\pi : M' \rightarrow M$ and define the vector bundle $VM' = \ker \pi_*$. A complementary distribution $HM'$ to $VM'$ in $TM'$ is called a nonlinear connection or a horizontal distribution on $M'$

$$TM' = HM' \oplus VM'.$$

A nonlinear connection $HM'$ enables us to define an almost complex structure on $M'$ as follows:

$$J : \Gamma(TM') \rightarrow \Gamma(TM'),$$

$$J \left( \frac{\delta}{\delta x^i} \right) = - \frac{\partial}{\partial y^i}, \quad J \left( \frac{\partial}{\partial y^i} \right) = \frac{\delta}{\delta x^i},$$

where $\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y) \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right\}$ is assumed as a local frame field of $TM'$ and $\Gamma(TM')$ is the space of smooth sections of the vector bundle $TM'$. We call $J$ the associate almost complex structure to $HM'$. Obviously we have $J^2 = -1d_{TM'}$, also we can assume the conjugate of $J$, $J' = -J$, as an almost complex structure. Now we give the following proposition which was proved by B.Y. Wu [12].

**Proposition 1.** Let $F^m = (M, M', F)$ be a Finsler manifold. Then the following statements are mutually equivalent:

1) $F^m = (M, M', F)$ has zero flag curvature;
2) $J$ is integrable;
3) $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection of the Sasaki metric $G$;
4) $(M', J, G)$ is Kählerian.

**Corollary 1.** Let the associate almost complex structure to $J$ (or $J'$) be a complex structure; then we have

$$\frac{\delta N^j_i}{\delta x^k} = \frac{\delta N^k_i}{\delta x^j}, \quad \frac{\partial N^j_i}{\partial y^k} = \frac{\partial N^k_i}{\partial y^j}.$$

So in this case the horizontal distribution is integrable.
3 \ (HM', S, T)-Cartan connection by using the associate almost complex structure \( J \)

In this section we give another way to define \((HM', S, T)\)-Cartan connection by using the associate almost complex structure \( J \) on \( M' \). Then we study the Kählerity of \((M', J, G)\), where \( G \) is the Sasaki metric and \( F^m = (M, M', F^*) \) is a Finsler manifold.

Let \( F^m = (M, M', F^*) \) be a pseudo-Finsler manifold. Then a Finsler connection on \( F^m \) is a pair \( FC = (HM', \nabla) \) where \( HM' \) is a nonlinear connection on \( M' \) and \( \nabla \) is a linear connection on the vertical vector bundle \( VM' \) (see \([1]\)).

**Theorem 1.** Let \( \nabla \) be a FC on \( M' \). The differential operator \( D \) defined by

\[
D_XY = \nabla_XvY - J\nabla_XJhY \quad \forall \ X, Y \in \Gamma(TM')
\]

is a linear connection on \( M' \). Also \( J \) is parallel with respect to \( D \), that is

\[
(D_XJ)Y = 0 \quad \forall \ X, Y \in \Gamma(TM').
\]

We call \( D \) the natural almost complex linear connection associated to FC \( \nabla \) on \( M' \).

**Proof.** For any \( X, Y, Z \in \Gamma(TM') \) and \( f \in C^\infty(M') \) we have

\[
D_{fX+YZ} = \nabla_{fX}vZ + \nabla_YvZ - J(f\nabla_XJhZ + \nabla_YJhZ)
\]

\[
= f(\nabla_XvZ - J\nabla_XJhZ) + \nabla_YvZ - J\nabla_YJhZ = fD_XZ + D_YZ,
\]

\[
D_X(fY + Z) = Xf(vY + hY) + f(\nabla_XvY - J\nabla_XJhY) + \nabla_XvZ - J\nabla_XJhZ
\]

\[
= (Xf)Y + fD_XY + D_XZ.
\]

Therefore \( D \) is a linear connection on \( M' \).

Also we have

\[
(D_XJ)(Z) = D_X(J(Z)) - J(D_XZ)
\]

\[
= \nabla_Xv(J(Z)) - J\nabla_XJ(h(J(Z))) - J\nabla_XvZ - \nabla_XJhZ
\]

\[
= \nabla_X\left(-Z^i \frac{\partial}{\partial y^i}\right) - J\nabla_X\left(-\tilde{Z}^i \frac{\partial}{\partial y^i}\right) - J\nabla_X\left(\tilde{Z}^i \frac{\partial}{\partial y^i}\right) - \nabla_X\left(-Z^i \frac{\partial}{\partial y^i}\right) = 0,
\]

where in local coordinates \( Z = Z^i \frac{\partial}{\partial x^i} + \tilde{Z}^i \frac{\partial}{\partial y^i} \).

Note that the torsion of \( D \) is given by the following expression:

\[
T^D(X, Y) = (\nabla_XvY - \nabla_YvX - v[X, Y]) - J(\nabla_XJhY - \nabla_YJhX - Jh[X, Y]).
\]

**Theorem 2.** Let \( HM' \) be a nonlinear connection on \( M' \) and \( S \) and \( T \) be any two skew-symmetric Finsler tensor fields of type \((1, 2)\) on \( F^m \). Then there exists a unique linear connection \( \nabla \) on \( VM' \) satisfying the conditions:

(i) \( \nabla \) is a metric connection;

(ii) \( T^D, S \) and \( T \) satisfy

\[
(a) \quad T^D(vX, vY) = S(vX, vY), \quad (b) \quad hT^D(hX, hY) = JT(JhX, JhY)
\]

for any \( X, Y \in \Gamma(TM') \), where \( J \) is the associate almost complex structure to \( HM' \).
Proof. This proof is similar to [1]. We define a linear connection $\nabla$ on $VM'$ by using $g$, $h$, $v$, $J$, $S$ and $T$ in the following way. For any $X, Y, Z \in \Gamma(TM')$ let

$$2g(\nabla_{vX}vY, vZ) = vX(g(vY, vZ)) + vY(g(vZ, vX)) - vZ(g(vX, vY))$$

$$+ g(vY, [vZ, vX]) + g(vZ, [vX, vY]) - g(vX, [vY, vZ]) + g(vY, S(vZ, vX))$$

$$+ g(vZ, S(vX, vY)) - g(vX, S(vY, vZ))$$

and

$$2g(\nabla_{hX}JhY, JhZ) = hX(g(JhY, JhZ)) + hY(g(JhZ, JhX))$$

$$- hZ(g(JhX, JhY)) + g(JhY, Jh[hZ, hX]) + g(JhZ, Jh[hX, hY])$$

$$- g(JhX, Jh[hY, hZ]) + g(JhY, T(JhZ, JhX))$$

$$+ g(JhZ, T(JhX, JhY)) - g(JhX, T(JhY, JhZ)).$$

Then for any $X, Y, Z \in \Gamma(TM')$ we have

$$(\nabla_x g)(vY, vZ) = (\nabla_{vX + hX} g)(vY, vZ)$$

$$= vX(g(vY, vZ)) - g(\nabla_{vX} vY, vZ) - g(vY, \nabla_{vX} vZ) + hX(g(vY, vZ))$$

$$- g(\nabla_{hX} vY, vZ) - g(vY, \nabla_{hX} vZ) = 0.$$

The above computation shows that the connection $\nabla$ defined by (2) and (3) is a metric connection.

Locally we set $\nabla = \frac{\partial}{\partial x^i}$, $\nabla \frac{\partial}{\partial y^j} = C^k_{ij} \frac{\partial}{\partial y^k}$, $S(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = S^h_{ij} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k}$ and $T(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = T^h_{ij} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k}$.

Now in (2) let $X = \frac{\partial}{\partial y^j}$, $Y = \frac{\partial}{\partial y^k}$ and $Z = \frac{\partial}{\partial y^l}$. After performing some computations we obtain the following expression for the coefficients $C^m_{ij}$:

$$C^m_{ij} = \frac{1}{2} \left\{ \frac{\partial g_{ij}}{\partial y^j} + \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ij}}{\partial y^k} + S^h_{ij} g_{ih} + S^h_{ij} g_{ih} - S^h_{li} g_{jh} \right\} g^{jm}.$$

Also in (3) let $X = \frac{\delta}{\delta x^i}$, $Y = \frac{\delta}{\delta x^j}$ and $Z = \frac{\delta}{\delta x^l}$. Then we can obtain the following expression for the coefficients $F^m_{ij}$:

$$F^m_{ij} = \frac{1}{2} \left\{ \frac{\delta g_{ij}}{\delta x^j} + \frac{\delta g_{ij}}{\delta x^k} - \frac{\delta g_{ij}}{\delta x^l} - T^h_{ij} g_{ih} - T^h_{ij} g_{ih} + T^h_{li} g_{jh} \right\} g^{jm}.$$

By using the relations $J \circ v = h \circ J$, $v \circ J = J \circ h$ and (1) we have

$$T^D(vX, vY) = \nabla_{vX}vY - \nabla_{vY}vX - [vX, vY],$$

$$hT^D(hX, hY) = J(\nabla_{hY}.JhX - \nabla_{hX}hY + Jh[hX, hY]).$$

Suppose that $X, Y \in \Gamma(TM')$ are two arbitrary vector fields on $M'$. In local coordinates, let $X = X^i \frac{\delta}{\delta x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ and $Y = Y^i \frac{\delta}{\delta x^i} + \tilde{Y}^i \frac{\partial}{\partial y^i}$, after performing some computations we have:

$$T^D \left( \tilde{X}^i \frac{\partial}{\partial y^i}, \tilde{Y}^i \frac{\partial}{\partial y^i} \right) = S \left( \tilde{X}^i \frac{\partial}{\partial y^i}, \tilde{Y}^i \frac{\partial}{\partial y^i} \right),$$

$$hT^D \left( X^i \frac{\delta}{\delta x^i}, Y^i \frac{\delta}{\delta x^i} \right) = JT \left( J \left( X^i \frac{\delta}{\delta x^i} \right), J \left( Y^i \frac{\delta}{\delta x^i} \right) \right).$$

The relations (6) and (7) show that $\nabla$ satisfies (ii) of Theorem 2.
Now let $\tilde{\nabla}$ be another linear connection on $VM'$ which satisfies (i) and (ii). By using the relations (i), (ii), (4) and (5) for $\tilde{\nabla}$ we have the following expressions:

$$vX(g(vY, vZ)) + vY(g(vZ, vX)) - vZ(g(vX, vY))$$

$$= g(\tilde{\nabla}_vXvY + \tilde{\nabla}_vXvY - T^D(vX, vY) - [vX, vY], vZ)$$

$$+ g(T^D(vX, vY) + [vX, vZ], vY) + g(T^D(vY, vZ) + [vY, vZ], vX),$$

$$hX(g(vJY, vJZ)) + hY(g(vZ, vJX)) - hZ(g(vJX, vJY))$$

$$= g(\tilde{\nabla}_{hX}JhY + \tilde{\nabla}_{hX}JhY - JT(JhX, JhY) - Jh[hX, hY], JhZ)$$

$$+ g(JT(JhX, JhZ) + Jh[hX, hZ], JhY) + g(JT(JhY, JhZ) + Jh[hY, hZ], JhX).$$

The relations (8) and (9) show that $\tilde{\nabla}$ satisfies (2) and (3), respectively. Therefore $\nabla = \tilde{\nabla}$. ■

The Finsler connection $FC = (HM', \nabla)$ where $\nabla$ is given by Theorem 2 is called the $(HM', S, T)$-Cartan connection (see [1, 2]) which in this case is obtained by the associate almost complex structure to $HM'$. If, in particular, $HM'$ is just the canonical nonlinear connection $GM'$ of $\mathbb{F}^m$ (for more details about $GM'$ see [1]) and $S = T = 0$, the $FC$ is called the Cartan connection and it is denoted by $FC^* = (GM', \nabla^*)$.

By means of the pseudo-Riemannian metric $g$ on $VM'$ we consider a pseudo-Riemannian metric on the vector bundle $TM'$ similar to the Sasaki one and denote it by $G$, that is

$$G = g_{ij}(x, y)dx^i dx^j + g_{ij}(x, y)\delta y^i \delta y^j,$$

where $\delta y^i = dy^i + N^{i}_{j}(x, y)dx^j$. Denote by $\nabla'$ the Levi-Civita connection on $M'$ with respect to $G$. A. Bejancu and H.R. Farran showed $\nabla^*$ is the projection of the Levi-Civita connection $\nabla'$ on the vertical vector bundle also they proved the following theorem (see [1]).

**Theorem 3.** The associate linear connection $D^*$ to the Cartan connection $FC^* = (GM', \nabla^*)$ is a metric linear connection with respect to $G$.

Now we give the following theorem which shows the natural almost complex linear connections associated to $(HM', S, T)$-Cartan connections are metric linear connections with respect to $G$.

**Theorem 4.** The natural almost complex linear connection $D$ associated to a $(HM', S, T)$-Cartan connection $FC = (HM', \nabla)$ is a metric linear connection with respect to $G$.

**Proof.** For any $X, X_1, X_2 \in \Gamma(TM')$ we have

$$D_XG(X_1, X_2) = XG(X_1, X_2) - G(D_XX_1, X_2) - G(X_1, D_XX_2)$$

$$= X(G(X_1, X_2)) - G(\nabla_XvX_1, X_2) + G(J\nabla_XJhX_1, X_2) - G(X_1, \nabla_XvX_2) + G(X_1, J\nabla_XJhX_2).$$

(10)

By (10) and this fact that $S$ and $T$ are skew-symmetric we have:

$$D_{\delta^i_y}G \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k} \right) = D_{\delta^i_x}G \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k} \right) = 0,$$

$$D_{\delta^j_x}G \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k} \right) = D_{\delta^j_y}G \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k} \right) = \frac{\partial g_{jk}}{\partial y^i} - C^{h}_{jh}g_{hk} - C^{h}_{ki}g_{jh} = 0,$$

$$D_{\delta^k_x}G \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = D_{\delta^k_y}G \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = \frac{\partial g_{jk}}{\partial x^i} - F^{h}_{ij}g_{hk} - F^{h}_{ki}g_{jh} = 0.$$

Therefore $D_XG = 0$ for any $X \in \Gamma(TM')$. ■
Let $F^m = (M, M', F)$ be a Finsler manifold. We can easily check that the pair $(J, G)$ defines an almost Hermitian metric on $M'$. In the following theorem we give a sufficient condition for Finsler tensor fields $S$ and $T$ such that $\mathcal{D}$ be the Levi-Civita connection arising from $G$.

**Theorem 5.** The natural almost complex linear connection $\mathcal{D}$ associated to a $(HM', S, T)$-Cartan connection $FC = (HM', \nabla)$ is the Levi-Civita connection arising from $G$ if $T^D(X, Y) = 0$ for any $X, Y \in \Gamma(TM')$ or equivalently if

\[
S = T = 0, \quad C_{ij}^k = R_{ij}^k = 0, \quad F_{ik} = \frac{\partial N_{ij}^k}{\partial y^i},
\]

where $R_{ij}^k = \frac{\delta N_{ij}^k}{\delta x^j} - \frac{\delta N_{ij}^k}{\delta x^i}$.

**Proof.** By Theorem 4, $\mathcal{D}$ is a metric linear connection with respect to $G$. Therefore if $T^D = 0$ then $\mathcal{D}$ is the Levi-Civita connection. In local coordinates we have

\[
T^D \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = S_{ij}^k \frac{\partial}{\partial y^k},
\]

\[
T^D \left( \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right) = C_{ji}^k \frac{\delta}{\delta x^k} + \left( \frac{\partial N_{ij}^k}{\partial y^j} - F_{ij}^k \right) \frac{\partial}{\partial y^k},
\]

\[
T^D \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = T_{ij}^k \frac{\delta}{\delta x^k} + \left( \frac{\delta N_{ij}^k}{\delta x^i} + \frac{\delta N_{ij}^k}{\delta x^j} \right) \frac{\partial}{\partial y^k}.
\]

Therefore the proof is completed.

**Corollary 2.** If $T^D = 0$ then $(M', J, G)$ is a Kähler manifold.

**Proof.** If $T^D = 0$ then $\mathcal{D}$ is the Levi-Civita connection of $G$. Also $J$ is parallel with respect to $\mathcal{D}$. Therefore $\mathcal{D}$ (the Levi-Civita connection of $G$) is almost complex. Consequently by using Theorem 4.3 of [10], $(M', J, G)$ is a Kähler manifold.

We know that the almost Hermitian manifold $(M', J, G)$ is an almost Kähler manifold if and only if the fundamental 2-form $\Phi$ is closed ($\Phi$ is defined by $\Phi(X, Y) = G(X, JY)$ for all $X, Y \in \Gamma(TM')$). Therefore we can give the following theorem.

**Theorem 6.** The almost Hermitian manifold $(M', J, G)$ is an almost Kähler manifold if and only if

\[
\frac{\delta g_{ik}}{\delta x^j} + \frac{\partial N_{ik}^h}{\partial y^j} g_{hj} - \left( \frac{\delta g_{ij}}{\delta x^k} + \frac{\partial N_{ij}^h}{\partial y^j} g_{hk} \right) = 0
\]

(11)

and

\[
R_{ij}^h g_{hk} - R_{ik}^h g_{hj} + R_{jk}^h g_{hi} = 0.
\]

(12)

**Proof.** Let $X_0, X_1, X_2 \in \Gamma(TM')$. Then we have

\[
d\Phi(X_0, X_1, X_2) = X_0 G(X_1, JX_2) - X_1 G(X_0, JX_2) + X_2 G(X_0, JX_1) - G([X_0, X_1], JX_2) + G([X_0, X_2], JX_1) - G([X_1, X_2], JX_0).
\]

By using the above relation in local coordinates we have:

\[
d\Phi \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \right) = d\Phi \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k} \right) = 0,
\]
The Relation Between the Associate Almost Complex Structure

\[ d\Phi\left(\frac{\partial}{\partial y^i}, \delta \frac{\partial}{\partial x^j}, \delta \frac{\partial}{\partial x^k}\right) = \delta g_{ik}^l \frac{\partial}{\partial x^j} + \frac{\partial N^h_k}{\partial y^l} g_{lj} \delta \frac{\partial}{\partial y^i} - \left(\delta g_{ij}^k \delta x^k + \frac{\partial N^h_i}{\partial y^k} g_{hk}\right), \]

\[ d\Phi\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = \left(\frac{\delta N^h_i}{\delta x^j} - \frac{\delta N^h_j}{\delta x^i}\right) g_{ik} - \left(\frac{\delta N^h_i}{\delta x^k} - \frac{\delta N^h_k}{\delta x^i}\right) g_{hj} + \left(\frac{\delta N^h_j}{\delta x^k} - \frac{\delta N^h_k}{\delta x^j}\right) g_{hi}. \]

Therefore the fundamental 2-form \( \Phi \) is closed if and only if the equations (11) and (12) are confirmed.

Now, by using Proposition 1 and Corollary 2, we have the following corollary.

**Corollary 3.** Let \( F^m = (M, M', F) \) be a Finsler manifold. If \( T^D = 0 \) then,

1) \( F^m = (M, M', F) \) has zero flag curvature;
2) \( J \) is integrable;
3) \( \nabla J = 0 \), where \( \nabla \) is the Levi-Civita connection of the Sasaki metric \( G \);
4) \( (M', J, G) \) is Kählerian.

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