A Banach Principle for Semifinite von Neumann Algebras

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Abstract. Utilizing the notion of uniform equicontinuity for sequences of functions with the values in the space of measurable operators, we present a non-commutative version of the Banach Principle for \( L^\infty \).

Key words: von Neumann algebra; measure topology; almost uniform convergence; uniform equicontinuity; Banach Principle

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1 Introduction

Let \((\Omega, \Sigma, \mu)\) be a probability space. Denote by \( L = L(\Omega, \mu) \) the set of all (classes of) complex-valued measurable functions on \( \Omega \). Let \( \tau_\mu \) stand for the measure topology in \( L \). The classical Banach Principle may be stated as follows.

Classical Banach Principle. Let \((X, \| \cdot \|)\) be a Banach space, and let \( a_n : (X, \| \cdot \|) \to (L, \tau_\mu) \) be a sequence of continuous linear maps. Consider the following properties:

(I) the sequence \( \{a_n(x)\} \) converges almost everywhere (a.e.) for every \( x \in X \);

(II) \( a^*(x)(\omega) = \sup_n |a_n(x)(\omega)| < \infty \) a.e. for every \( x \in X \);

(III) (II) holds, and the maximal operator \( a^* : (X, \| \cdot \|) \to (L, \tau_\mu) \) is continuous at 0;

(IV) the set \( \{x \in X : \{a_n(x)\} \text{ converges a.e.}\} \) is closed in \( X \).

Implications (I) \( \Rightarrow \) (II) \( \Rightarrow \) (III) \( \Rightarrow \) (IV) always hold. If, in addition, there is a set \( D \subset X \), \( D = X \), such that the sequence \( \{a_n(x)\} \) converges a.e. for every \( x \in D \), then all four conditions (I)–(IV) are equivalent.

The Banach Principle is most often and successfully applied in the context \( X = (L^p, \| \cdot \|_p) \), \( 1 \leq p < \infty \). At the same moment, in the case \( p = \infty \) the uniform topology in \( L^\infty \) appears to be too strong for the “classical” Banach Principle to be effective in \( L^\infty \). For example, continuous functions are not uniformly dense in \( L^\infty \).

In [1], employing the fact that the unit ball \( L^\infty_1 = \{x \in L^\infty : \|x\|_\infty \leq 1\} \) is complete in \( \tau_\mu \), the authors suggest to consider the measure topology in \( L^\infty \) replacing \( (X, \| \cdot \|) \) by \( (L^\infty_1, \tau_\mu) \). Note that, since \( L^\infty_1 \) is not a linear space, geometrical complications occur, which in [1] are treated with the help of the following lemma.

Lemma 1. If \( N(x, \delta) = \{y \in L^\infty_1 : \|y - x\|_1 \leq \delta\} \), \( x \in L^\infty_1 \), \( \delta > 0 \), then \( N(0, \delta) \subset N(x, \delta) - N(x, \delta) \) for any \( x \in L^\infty_1 \), \( \delta > 0 \).
An application of the Baire category theorem yields the following replacement of \((I) \Rightarrow (II)\).

**Theorem 1** ([1]). Let \(a_n : L^\infty \to \mathcal{L}\) be a sequence of \(\tau_\mu\)-continuous linear maps such that the sequence \(\{a_n(x)\}\) converges a.e. for all \(x \in L^\infty\). Then the maximal operator \(a^*(\cdot) = \sup_n |a_n(x)(\omega)|, x \in L^\infty\), is \(\tau_\mu\)-continuous at 0 on \(L^1_\infty\).

At the same time, as it is known [1], even for a sequence \(a_n : L^\infty \to L^\infty\) of contractions, in which case condition (II) is clearly satisfied, the maximal operator \(a^* : L^1_\infty \to L^1_\infty\) may be not \(\tau_\mu\)-continuous at 0, i.e., (II) does not necessarily imply (III), whereas a replacement of the implication (III) \(\Rightarrow (IV)\) does hold:

**Theorem 2** ([1]). Assume that each \(a_n : L^\infty \to \mathcal{L}\) is linear, condition (II) holds with \(X = L^\infty\), and the maximal operator \(a^* : L^\infty \to \mathcal{L}\) is \(\tau_\mu\)-continuous at 0 on \(L^1_\infty\). Then the set \(\{x \in L^1_\infty : \{a_n(x)\}\ \text{converges a.e.}\}\) is closed in \((L^1_\infty, \tau_\mu)\).

A non-commutative Banach Principle for measurable operators affiliated with a semifinite von Neumann algebra was established in [5]. Then it was refined and applied in [7, 4, 3]. In [3] the notion of uniform equicontinuity of a sequence of functions into \(L(M, \tau)\) was introduced. The aim of this study is to present a non-commutative extension of the Banach Principle for \(L^\infty\) that was suggested in [1]. We were unable to prove a verbatim operator version of Lemma 1. Instead, we deal with the mentioned geometrical obstacles via essentially non-commutative techniques, which helps us to get rid of some restrictions in [1]. First, proof of Lemma 1 essentially depends on the assumption that the functions in \(\mathcal{L}\) be real-valued while the argument of the present article does not employ this condition. Also, our approach eliminates the assumption of the finiteness of measure.

## 2 Preliminaries

Let \(M\) be a semifinite von Neumann algebra acting on a Hilbert space \(H\), and let \(B(H)\) denote the algebra of all bounded linear operators on \(H\). A densely-defined closed operator \(x\) in \(H\) is said to be affiliated with \(M\) if \(y'x \subset xy'\) for every \(y' \in B(H)\) with \(y'z = zy', z \in M\). We denote by \(P(M)\) the complete lattice of all projections in \(M\). Let \(\tau\) be a faithful normal semifinite trace on \(M\). If \(I\) is the identity of \(M\), denote \(e^\perp = I - e, e \in P(M)\). An operator \(x\) affiliated with \(M\) is said to be \(\tau\)-measurable if for each \(\epsilon > 0\) there exists a projection \(e \in P(M)\) with \(\tau(e^\perp) \leq \epsilon\) such that \(eH\) lies in the domain of the operator \(x\). Let \(L = L(M, \tau)\) stand for the set of all \(\tau\)-measurable operators affiliated with \(M\). Denote \(\| \cdot \|\) the uniform norm in \(B(H)\). If for any given \(\epsilon > 0\) and \(\delta > 0\) one sets

\[
V(\epsilon, \delta) = \{x \in L : \|xe\| \leq \delta \text{ for some } e \in P(M) \text{ with } \tau(e^\perp) \leq \epsilon\},
\]

then the topology \(t_\tau\) in \(L\) defined by the family \(\{V(\epsilon, \delta) : \epsilon > 0, \delta > 0\}\) of neighborhoods of zero is called a measure topology.

**Theorem 3** ([9], see also [8]). \((L, t_\tau)\) is a complete metrizable topological \(*\)-algebra.

**Proposition 1.** For any \(d > 0\), the sets \(M_d = \{x \in M : \|x\| \leq d\}\) and \(M^b_d = \{x \in M_d : x^* = x\}\) are \(t_\tau\)-complete.

**Proof.** Because \((L, t_\tau)\) is a complete metric space, it is enough to show that \(M_d\) and \(M^b_d\) are (sequentially) closed in \((L, t_\tau)\). If \(M_d \ni x_n \to_{t_\tau} x \in L\), then \(0 \leq x_n^*x_n \leq d \cdot I\) and, due to Theorem 3, \(x_nx_n^* \to_{t_\tau} x^*x\). Since \(\{x \in L : x \geq 0\}\) is \(t_\tau\)-complete, we have \(0 \leq x^*x \leq d \cdot I\), which implies that \(x \in M_d\). Therefore, \(M_d\) is closed in \((L, t_\tau)\). Similarly, it can be checked that \(M^b_d\) is closed in \((L, t_\tau)\).

\[\square\]
A sequence \( \{y_n\} \subset L \) is said to converge \textit{almost uniformly} (a.u.) to \( y \in L \) if for any given \( \epsilon > 0 \) there exists a projection \( e \in P(M) \) with \( \tau(e^+) \leq \epsilon \) satisfying \( \|(y - y_n)e\| \to 0 \).

**Proposition 2.** If \( \{y_n\} \subset L \), then the conditions

(i) \( \{y_n\} \) converges a.u. in \( L \);
(ii) for every \( \epsilon > 0 \) there exists \( e \in P(M) \) with \( \tau(e^+) \leq \epsilon \) such that \( \|(y_m - y_n)e\| \to 0 \) as \( m, n \to \infty \).

are equivalent.

**Proof.** Implication (i) \( \Rightarrow \) (ii) is trivial. (ii) \( \Rightarrow \) (i): Condition (ii) implies that the sequence \( \{y_n\} \) is fundamental in measure. Therefore, by Theorem 3, one can find \( y \in L \) such that \( y_n \to y \) in \( t_\tau \). Fix \( \epsilon > 0 \), and let \( p \in P(M) \) be such that \( \tau(p^+) \leq \epsilon/2 \) and \( \|(y_m - y_n)p\| \to 0 \) as \( m, n \to \infty \). Because the operators \( y_n, n \geq 1 \), are measurable, it is possible to construct such a projection \( q \in P(M) \) with \( \tau(q^+) \leq \epsilon/2 \) that \( \{y_nq\} \subset M \). Defining \( e = p \wedge q \), we obtain \( \tau(e^+) \leq \epsilon \), \( y_n e = y_n q e \in M \), and

\[
\|y_m e - y_n e\| = \|(y_m - y_n)p e\| \leq \|(y_m - y_n)p\| \to 0,
\]

\( m, n \to \infty \). Thus, there exists \( y(e) \in M \) satisfying \( \|y_n e - y(e)\| \to 0 \). In particular, \( y_n e \to y(e) \) in \( t_\tau \). On the other hand, \( y_n e = y e \) in \( t_\tau \), which implies that \( y(e) = y e \). Hence, \( \|(y_n - y)e\| \to 0 \), i.e. \( y_n \to y \) a.u. \( \blacksquare \)

The following is a non-commutative Riesz’s theorem [9]; see also [5].

**Theorem 4.** If \( \{y_n\} \subset L \) and \( y = t_\tau - \lim_{n \to \infty} y_n \), then \( y = a.u. - \lim_{k \to \infty} y_{n_k} \) for some subsequence \( \{y_{n_k}\} \subset \{y_n\} \).

### 3 Uniform equicontinuity for sequences of maps into \( L(M, \tau) \)

Let \( E \) be any set. If \( a_n : E \to L, x \in E, \) and \( b \in M \) are such that \( \{a_n(x)b\} \subset M \), then we denote

\[
S(x, b) = S(\{a_n\}, x, b) = \sup_n \|a_n(x)b\|.
\]

Definition below is in part due to the following fact.

**Lemma 2.** Let \( (X, +) \) be a semigroup, \( a_n : X \to L \) be a sequence of additive maps. Assume that \( \bar{x} \in X \) is such that for every \( \epsilon > 0 \) there exist a sequence \( \{x_k\} \subset X \) and a projection \( p \in P(M) \) with \( \tau(p^+) \leq \epsilon \) such that

(i) \( \{a_n(\bar{x} + x_k)\} \) converges a.u. as \( n \to \infty \) for every \( k \);
(ii) \( S(x_k, p) \to 0, k \to \infty \).

Then the sequence \( \{a_n(\bar{x})\} \) converges a.u. in \( L \).

**Proof.** Fix \( \epsilon > 0 \), and let \( \{x_k\} \subset X \) and \( p \in P(M), \tau(p^+) \leq \epsilon/2 \), be such that conditions (i) and (ii) hold. Pick \( \delta > 0 \) and let \( k_0 = k_0(\delta) \) be such that \( S(x_{k_0}, p) \leq \delta/3 \). By Proposition 2, there is a projection \( q \in P(M) \) with \( \tau(q^+) \leq \epsilon/2 \) and a positive integer \( N \) for which the inequality

\[
\|(a_n(\bar{x} + x_{k_0}) - a_n(\bar{x} + x_{k_0}))q\| \leq \frac{\delta}{3}
\]

holds whenever \( m, n \geq N \). If one defines \( e = p \wedge q \), then \( \tau(e^+) \leq \epsilon \) and

\[
\|(a_n(\bar{x}) - a_n(\bar{x}))e\| \leq \|(a_n(\bar{x} + x_{k_0}) - a_n(\bar{x} + x_{k_0}))e\| + \|a_n(x_{k_0})e\| + \|a_n(x_{k_0})e\| \leq \delta
\]

for all \( m, n \geq N \). Therefore, by Proposition 2, the sequence \( \{a_n(\bar{x})\} \) converges a.u. in \( L \). \( \blacksquare \)
Let \((X,t)\) be a topological space, and let \(a_n : X \to L\) and \(x_0 \in X\) be such that \(a_n(x_0) = 0\), \(n = 1, 2, \ldots\). Recall that the sequence \(\{a_n\}\) is equicontinuous at \(x_0\) if, given \(\epsilon > 0\) and \(\delta > 0\), there is a neighborhood \(U\) of \(x_0\) in \((X,t)\) such that \(a_n(U) \subset V(\epsilon,\delta), n = 1, 2, \ldots\), i.e., for every \(x \in U\) and every \(n\) one can find a projection \(e = e(x,n) \in P(M)\) with \(\tau(e^\perp) \leq \epsilon\) satisfying 
\[\|a_n(x)e\| \leq \delta.\]

**Definition.** Let \((X,t), a_n : X \to L\), and \(x_0 \in X\) be as above. Let \(x_0 \in E \subset X\). The sequence \(\{a_n\}\) will be called uniformly equicontinuous at \(x_0\) on \(E\) if, given \(\epsilon > 0, \delta > 0\), there is a neighborhood \(U\) of \(x_0\) in \((X,t)\) such that for every \(x \in E \cap U\) there exists a projection \(e = e(x) \in P(M)\), \(\tau(e^\perp) \leq \epsilon\), satisfying \(S(x,e) \leq \delta\).

As it can be easily checked, the uniform equicontinuity is a non-commutative generalization of the continuity of the maximal operator, a number of equivalent forms of which are presented in [1].

Let \(\rho\) be an invariant metric in \(L\) compatible with \(t_\tau\) (see Theorem 3).

**Lemma 3.** Let \(d > 0\). If a sequence \(a_n : M \to L\) of additive maps is uniformly equicontinuous at 0 on \(M_d^h\), then it is also uniformly equicontinuous at 0 on \(M_d\).

**Proof.** Fix \(\epsilon > 0, \delta > 0\). Let \(\gamma > 0\) be such that, given \(x \in M_d^h, \rho(0, x) < \gamma\), there is \(e = e(x) \in P(M)\) for which \(\tau(e^\perp) \leq \epsilon/2\) and \(S(x,e) \leq \delta/2\) hold. Pick \(x \in M_d\) with \(\rho(0, x) < \gamma\). We have \(x = \text{Re}(x) + i \text{Im}(x)\), where \(\text{Re}(x) = \frac{x + x^*}{2}, \text{Im}(x) = \frac{x - x^*}{2i}\). Clearly, \(\text{Re}(x), \text{Im}(x) \in M_d^h\) and \(\rho(0, \text{Re}(x)) < \gamma, \rho(0, \text{Im}(x)) < \gamma\). Therefore, one can find such \(p, q \in P(M)\) with \(\tau(p^\perp) \leq \epsilon/2\) and \(\tau(q^\perp) \leq \epsilon/2\) that \(S(\text{Re}(x), p) \leq \delta/2\) and \(S(\text{Im}(x), q) \leq \delta/2\). Defining \(r = p \wedge q\), we get \(\tau(r^\perp) \leq \epsilon\) and

\[S(x,r) \leq S(\text{Re}(x), r) + S(\text{Im}(x), r) \leq S(\text{Re}(x), p) + S(\text{Im}(x), q) \leq \delta,\]

implying that the sequence \(\{a_n\}\) is uniformly equicontinuous at 0 on \(M_d\). \(\blacksquare\)

**Lemma 4.** Let a sequence \(a_n : M \to L\) of additive maps be uniformly equicontinuous at 0 on \(M_d\) for some \(0 < d \in \mathbb{Q}\). Then \(\{a_n\}\) is also uniformly equicontinuous at 0 on \(M_s\) for every \(0 < s \in \mathbb{Q}\).

**Proof.** Pick \(0 < s \in \mathbb{Q}\), and let \(r = d/s\). Given \(\epsilon > 0, \delta > 0\), one can present such \(\gamma > 0\) that for every \(x \in M_d\) with \(\rho(0, x) < \gamma r\) there is a projection \(e = e(x) \in P(M)\), \(\tau(e^\perp) \leq \epsilon\), satisfying \(S(x,e) \leq \delta r\). Since \(a_n\) is additive and \(d, s \in \mathbb{Q}\), we have \(a_n(rx) = ra_n(x)\). Also, \(rx \in M_d\) and \(\rho(0, rx) < \gamma r\) is equivalent to \(x \in M_s\) and \(\rho(0, x) < \gamma\). Thus, given \(x \in M_s\) with \(\rho(0, x) < \gamma\), we have

\[\|a_n(x)e\| = \frac{1}{r} \cdot \|a_n(rx)e\| \leq \delta,\]

meaning that the sequence \(\{a_n\}\) is uniformly equicontinuous at 0 on \(M_s\). \(\blacksquare\)

### 4 Main results

Let \(0 \in E \subset M\). For a sequence of functions \(a_n : (M,t_\tau) \to L\), consider the following conditions

- (CNV(E)) almost uniform convergence of \(\{a_n(x)\}\) for every \(x \in E\);
- (CNT(E)) uniform equicontinuity at 0 on \(E\);
- (CLS(E)) closedness in \((E,t_\tau)\) of the set \(C(E) = \{x \in E : \{a_n(x)\} \text{ converges a.u.}\}\).
In this section we will study relationships among the conditions \((\text{CNV}(M_1)), \ (\text{CNT}(M_1)), \text{ and} \ (\text{CLS}(M_1)).

**Remarks.** 1. Following the classical scheme (see Introduction), one more condition can be added to this list, namely, a non-commutative counterpart of the existence of the maximal operator, which can be stated as [5]:

\[(\text{BND}(E)) \text{ given } x \in E \text{ and } e > 0, \text{ there is } e \in P(M), \tau(e) \leq \epsilon, \text{ with } S(x, e) < \infty.\]

This condition can be called a pointwise uniform boundedness of \(\{a_n\}\) on \(E\). It can be easily verified that \((\text{CNV}(E)) \text{ implies } (\text{BND}(E)).\) But, as it was mentioned in Introduction, even in the commutative setting, \((\text{BND}(M_1)) \text{ does not guarantee } (\text{CNT}(M_1)).\)

2. If \(a_n \text{ is additive for every } n\), then \((\text{CNV}(M)) \text{ follows from } (\text{CNV}(M_1)).\)

3. If \(E\) is closed in \((M, t_r)\) (for instance, if \(E = M_\theta\), or \(E = M_\theta^*; \text{ see Proposition 1}), \text{ then } (\text{CLS}(E)) \text{ is equivalent to the closedness of } C(E) \text{ in } (L, t_r).\)

In order to show that \((\text{CNV}(M_1)) \text{ entails } (\text{CNT}(M_1)), \text{ we will provide some auxiliary facts.}\)

**Lemma 5.** For any \(0 \leq x \in L \text{ and } e \in P(M), \ x \leq 2(e x e + e^\perp x e^\perp).\)

**Proof.** If \(a = e - e^\perp\), then \(a^* = a\), which implies that

\[0 \leq a x a = e x e - e^\perp x e + e^\perp x e^\perp.\]

Therefore, \(e x e + e^\perp x e \leq e x e + e^\perp x e^\perp\), and we obtain

\[x = (e + e^\perp) x (e + e^\perp) \leq 2(e x e + e^\perp x e^\perp).\]

For \(y \in M\), denote \(l(y)\) the projection on \(\overline{yH}\), and let \(r(y) = I - n(y), \text{ where } n(y) \text{ denotes }\) the projection on \(\{\xi \in H : y \xi = 0\}\). It is easily checked that \(l(y^*) = r(y)\), so, if \(y^* = y\), one can define \(s(y) = l(y) = r(y)\). The projections \(l(y), r(y), \text{ and } s(y)\) are called, respectively, a left support of \(y\), a right support of \(y\), and a support of \(y = y^*\). It is well-known that \(l(y)\) and \(r(y)\) are equivalent projections, in which case one writes \(l(y) \sim r(y)\). In particular, \(\tau(l(y)) = \tau(r(y))\), \(y \in M\). If \(y^* = y \in M, \ y_+ = \int_0^\infty \lambda dE_\lambda, \text{ and } y_- = -\int_{-\infty}^0 \lambda dE_\lambda\), where \(\{E_\lambda\}\) is the spectral family of \(y\), then we have \(y = y_+ - y_-\), \(y_+ = s(y_+)ys(y_+), \text{ and } y_- = -s(y_+)ys(y_+)^*\).

The next lemma is, in a sense, a non-commutative replacement of Lemma 0.1.

**Lemma 6.** Let \(y^* = y \in M, \ -I \leq y \leq I. \text{ Denote } e_+ = s(y_+). \text{ If } x \in M\text{ is such that } 0 \leq x \leq I, \text{ then}

\[-I \leq y - e_+ x e_+ \leq I \quad \text{ and } \quad -I \leq y + e_+^\perp x e_+^\perp \leq I.\]

**Proof.** Because \(e_+ x e_+ \geq 0\), we have \(y - e_+ x e_+ \leq y \leq I; \text{ analogously, } -I \leq y + e_+^\perp x e_+^\perp. \text{ On the other hand, since we obviously have } e_+ x e_+ \leq e_+, \ e_+^\perp x e_+^\perp \leq e_+, \ e_+ y e_+ \leq e_+, \text{ and } e_+^\perp y e_+^\perp \geq -e_+, \text{ one can write}

\[y - e_+ x e_+ = y_+ - y_- - e_+ x e_+ = y_+ + e_+^\perp y e_+^\perp - e_+ x e_+ \geq y_+ - e_+^\perp - e_+ = y_+ - I \geq -I\]

and

\[y + e_+^\perp x e_+^\perp = e_+ y e_+ - y_- + e_+^\perp x e_+^\perp \leq e_+ - y_- + e_+^\perp = I - y_- \leq I,\]

which finishes the proof.

**Lemma 7.** \(aV(e, \delta)b \subset V(2\epsilon, \delta) \text{ for all } \epsilon > 0, \delta > 0, \text{ and } a, b \in M_1.\)
Let \( x \in V(\epsilon, \delta) \). There exists \( e \in P(M) \) such that \( \tau(e^+) \leq \epsilon \) and \( \|xe\| \leq \delta \). If we denote \( q = n(e^+ b) \), then
\[
bq = (e + e^+)bq = ebq + e^+ bn(e^+ b) = ebq.
\]
Besides, we have \( q^+ = \tau(e^+ b) \sim I(e^+ b) \leq e^+ \), which implies that \( \tau(q^+) \leq \epsilon \). Now, if one defines \( p = e \land q \), then \( \tau(p^+) \leq 2\epsilon \) and
\[
\|axbp\| = \|axbqp\| = \|axebqp\| \leq \|axeb\| \leq \|a\| \cdot \|x\| \cdot \|b\| \leq \delta.
\]
Therefore, \( axb \in V(2\epsilon, \delta) \).

**Lemma 8 ([5]).** Let \( f \) be the spectral projection of \( b \in M \), \( 0 \leq b \leq I \), corresponding to the interval \([1/2, 1]\). Then
\begin{enumerate}[(i)]
  \item \( \tau(f^+) \leq 2 \cdot \tau(I - b) \);
  \item \( f = bc \) for some \( c \in M \) with \( 0 \leq c \leq 2 \cdot I \).
\end{enumerate}

We shall also need the following fundamental result.

**Theorem 5 ([6]).** Let \( a : M \to M \) be a positive linear map such that \( a(I) \leq I \). Then \( a(x^2) \leq a(x^2) \) for every \( x^* = x \in M \).

The next theorem represents a non-commutative extension of Theorem 1.

**Theorem 6.** Let \( a_n : M \to L \) be a (CNV(M)) sequence of positive \( t_\tau \)-continuous linear maps such that \( a_n(I) \leq I \), \( n = 1, 2, \ldots \). Then the sequence \( \{a_n\} \) is also (CNT(M)).

**Proof.** Fix \( \epsilon > 0 \) and \( \delta > 0 \). For \( N \in \mathbb{N} \) define
\[
F_N = \left\{ x \in M_1^b : \sup_{n \geq N} \|(a_N(x) - a_n(x))b\| \leq \delta \text{ for some } b \in M, 0 \leq b \leq I, \tau(I - b) \leq \epsilon \right\}.
\]
Show that the set \( F_N \) is closed in \( (M_1^b, \rho) \). Let \( \{y_m\} \subset F_N \) and \( \rho(y_m, \tilde{x}) \to 0 \) for some \( \tilde{x} \in L \). It follows from Proposition 1 that \( \tilde{x} \in M_1^b \). We have \( a_1(y_m) \to a_1(\tilde{x}) \) in \( t_\tau \), which, by Theorems 3 and 4, implies that there is a subsequence \( \{y_{m_k}^{(1)}\} \subset \{y_m\} \) such that \( a_1(y_{m_k}^{(1)}) \to a_1(\tilde{x})^* \) a.u. Similarly, there is a subsequence \( \{y_{m_k}^{(2)}\} \subset \{y_{m_k}^{(1)}\} \) for which \( a_2(y_{m_k}^{(2)})^* \to a_2(\tilde{x})^* \) a.u. Repeating this process and defining \( x_m = y_{m_k}^{(m)} \in F_N, m = 1, 2, \ldots \), we obtain
\[
a_n(x_m)^* \to a_n(\tilde{x})^* \text{ a.u., } m \to \infty, \quad n = 1, 2, \ldots.
\]

By definition of \( F_N \), there exists a sequence \( \{b_m\} \subset M, 0 \leq b_m \leq I, \tau(I - b_m) \leq \epsilon \), such that \( \sup_{n \geq N} \|(a_N(x_m) - a_n(x_m))b_m\| \leq \delta \) for every \( m \). Because \( M_1 \) is weakly compact, there are a subnet \( \{b_\alpha\} \subset \{b_m\} \) and \( b \in M \) such that \( b_\alpha \to b \) weakly, i.e. \( (b_\alpha \xi, \xi) \to (b \xi, \xi) \) for all \( \xi \in H \). Clearly \( 0 \leq b \leq I \). Besides, by the well-known inequality (see, for example [2]),
\[
\tau(I - b) \leq \liminf_\alpha \tau(I - b_\alpha) \leq \epsilon.
\]

We shall show that \( \sup_{n \geq N} \|(a_N(\tilde{x}) - a_n(\tilde{x}))b\| \leq \delta \). Fix \( n \geq N \). Since \( a_k(x_m)^* \to a_k(\tilde{x})^* \) a.u., \( k = n, N \), given \( \sigma > 0 \), there exists a projection \( e \in P(M) \) with \( \tau(e^+) \leq \sigma \) satisfying
\[
\|e(a_k(x_m) - a_k(\tilde{x}))\| = \|(a_k(x_m)^* - a_k(\tilde{x})^*)e\| \to 0, \quad m \to \infty, \quad k = n, N.
\]
Show first that \( \|e(a_N(\tilde{x}) - a_n(\tilde{x}))b\| \leq \delta \). For every \( \xi, \eta \in H \) we have
\[
\|(e(a_N(\tilde{x}) - a_n(\tilde{x}))b)m - (a_N(\tilde{x}) - a_n(\tilde{x}))b)\xi, \eta\|
\]
\[
\leq |(e(a_N(x_m) - a_n(x_m)) - a_N(x) + a_n(x))b_m\xi, \eta) + |((b_m - b)\xi, (a_N(x)^* - a_n(x)^*)\eta)|. \\
\]

Fix \( \gamma > 0 \) and choose \( m_0 \) be such that
\[
\|e(a_k(x_m) - a_k(x))\| < \gamma, \quad k = n, N
\]
whenever \( m \geq m_0 \). Since \( b_\alpha \to b \) weakly, one can find such an index \( \alpha(\gamma) \) that
\[
|((b_\alpha - b)\xi, (a_N(x)^* - a_n(x)^*)\eta)| < \gamma
\]
as soon as \( \alpha \geq \alpha(\gamma) \). Because \( \{b_\alpha\} \) is a subnet of \( \{b_m\} \), there is such an index \( \alpha(m_0) \) that \( \{b_\alpha\}_{\alpha=\alpha(m_0)} \subset \{b_m\}_{m=m_0} \). In particular, if \( \alpha_0 = \max \{\alpha(\gamma), \alpha(m_0)\} \), then \( b_{\alpha_0} = b_{m_0} = b \) for some \( m_1 \geq m_0 \). It follows now from (1)–(3) that, for all \( \xi, \eta \in H \) with \( \|\xi\| = \|\eta\| = 1 \), we have
\[
|e(a_N(x) - a_n(x))b\xi, \eta)| \leq |(e(a_N(x_m_1) - a_n(x_m_1))b_m\xi, \eta)| + |(e(a_{m_1}(x_{m_1}) - a_n(x) + a_n(x))b_{m_1}\xi, \eta)| + |((b_{m_1} - b)\xi, (a_N(x)^* - a_n(x)^*)\eta)|.
\]

Due to the arbitrariness of \( \gamma > 0 \), we get
\[
\|e(a_N(x) - a_n(x))b\| = \sup_{\|\xi\| = \|\eta\| = 1} |(e(a_N(x) - a_n(x))b\xi, \eta)| \leq \delta.
\]

Next, we choose \( e_j \in P(M) \) such that \( \tau(e_j) \leq \frac{1}{j} \) and
\[
\|e_j(a_k(x_m) - a_k(x))\| \to 0 \quad \text{as} \quad m \to \infty, \quad k = n, N; \quad j = 1, 2, \ldots.
\]

Since \( e_j \to I \) weakly, \( e_j(a_N(x) - a_n(x))b \to (a_N(x) - a_n(x))b \) weakly, therefore,
\[
\|(a_N(x) - a_n(x))b\| \leq \limsup_{j \to \infty} \|e_j(a_N(x) - a_n(x))b\| \leq \delta.
\]

Thus, for every \( n \geq N \) the inequality \( \|(a_N(x) - a_n(x))b\| \leq \delta \) holds, which implies that \( x \in F_N \) and \( F_N = F_N \).

Further, as \( \{a_n(x)\} \) converges a.u. for every \( x \in M_1 \), taking into account Proposition 2, we obtain
\[
M^b_1 = \bigcup_{N=1}^{\infty} F_N.
\]

By Proposition 1, the metric space \( (M^b_1, \rho) \) is complete. Therefore, using the Baire category theorem, one can present such \( \rho_0 \) that \( F_{\rho_0} \) contains an open set. In other words, there exist \( x_0 \in F_{\rho_0} \) and \( \gamma_0 \geq 0 \) such that for any \( x \in M^b_1 \) with \( \rho(x_0, x) < \gamma_0 \) it is possible to find \( b_x \in M \), \( 0 \leq b_x \leq I \), satisfying \( \tau(I - b_x) \leq \epsilon \) and
\[
\sup_{n \geq 0} \|(a_{\rho_0}(x) - a_n(x))b_x\| \leq \delta.
\]

Let \( f_{\rho} \) be the spectral projection of \( b_x \) corresponding to the interval \([1/2, 1]\). Then, according to Lemma 8, \( \tau(f_{\rho}^\perp) \leq 2\epsilon \) and
\[
\sup_{n \geq 0} \|(a_{\rho_0}(x) - a_n(x))f_{\rho}\| \leq 2\delta
\]
whenever $x \in M_1^h$ and $\rho(x_0, x) < \gamma_0$. Since the multiplication in $L$ is continuous with respect to the measure topology, Lemma 7 allows us to choose $0 < \gamma_1 < \gamma_0$ in such a way that $\rho(0, x) < \gamma_1$ would imply $\rho(0, ax^2b) < \gamma_0$ for every $a, b \in M_1$. Denote $e_+ = s(x_0^+)$. Because $a_i : (M, \rho) \to (L, t_\tau)$ is continuous for each $i$, there exists such $0 < \gamma_2 < \gamma_1$ that, given $x \in M$ with $\rho(0, x) < \gamma_2$, it is possible to find such a projection $p \in P(M)$, $\tau(p^\perp) \leq \epsilon$, that

$$\|a_i(e_+x^2e_+)p\| \leq \delta \quad \text{and} \quad \|a_i(e_+x^2e_+)\| \leq \delta,$$

so, we have

$$y = x_0 - e_+x^2e_+ \in M_1^h \quad \text{and} \quad z = x_0 + e_+x^2e_+ \in M_1^h.$$

Besides, $\rho(x_0, y) = \rho(0, -e_+x^2e_+) < \gamma_0$, which implies that there is $f_1 \in P(M)$ such that $\tau(f_1^\perp) \leq 2\epsilon$ and

$$\sup_{n \geq N_0} \|(a_{N_0}(y) - a_n(y))f_1\| \leq 2\delta.$$ 

Analogously, one finds $f_2 \in P(M), \tau(f_2^\perp) \leq 2\epsilon$, satisfying

$$\sup_{n \geq N_0} \|(a_{N_0}(z) - a_n(z))f_2\| \leq 2\delta.$$ 

As $\rho(0, x) < \gamma_2$, there is $p \in P(M)$ with $\tau(p^\perp) \leq \epsilon$ such that the inequalities

$$\|a_i(e_+x^2e_+)p\| \leq \delta \quad \text{and} \quad \|a_i(e_+x^2e_+)\| \leq \delta$$

hold for all $i = 1, \ldots, N_0$. Let $e = f_{x_0} \wedge f_1 \wedge f_2 \wedge p$. Then we have $\tau(e^\perp) \leq 7\epsilon$ and, for $n > N_0$,

$$\|a_n(e_+x^2e_+)e\| \leq \|(a_{N_0}(x_0 - e_+x^2e_+)) - a_n(x_0 - e_+x^2e_+) \leq \|a_{N_0}(y) - a_n(y))f_1e\|$$

$$+ \|a_{N_0}(x_0 - a_n(x_0))f_2xe\| + \|a_{N_0}(e_+x^2e_+)p\| \leq 5\delta.$$ 

At the same time, if $n \in \{1, \ldots, N_0\}$, then $\|a_n(e_+x^2e_+)e\| = \|a_n(e_+x^2e_+)pe\| \leq \delta$, so

$$\|a_n(e_+x^2e_+)e\| \leq 5\delta, \quad n = 1, 2, \ldots.$$ 

Analogously,

$$\|a_n(e_+x^2e_+)e\| \leq 5\delta, \quad n = 1, 2, \ldots.$$ 

Next, by Lemma 5, we can write $0 \leq x^2 \leq 2(e_+x^2e_+ + e_+x^2e_+).$ Since $a_n$ is positive for every $n$, applying Theorem 5, we obtain

$$0 \leq e a_n(x)^2 e \leq e a_n(x^2 e) \leq 2(e a_n(e_+x^2e_+ e) + e a_n(e_+x^2e_+ e).$$

Therefore,

$$\|a_n(x)e\|^2 = \|e a_n(x)^2 e\| \leq 20\delta, \quad n = 1, 2, \ldots.$$ 

Summarizing, given $\epsilon > 0$, $\delta > 0$, it is possible to find such $\gamma > 0$ that for every $x \in M_1^h$ with $\rho(0, x) < \gamma$ there is a projection $e = e(x) \in P(M)$ such that $\tau(e^\perp) \leq 7\epsilon$ and

$$S(x, e) = \sup_n \|a_n(x)e\| \leq \sqrt{20\delta}.$$ 

Thus, the sequence $\{a_n\}$ is $(\text{CNT}(M_1^h))$, hence, by Lemma 3, $(\text{CNT}(M_1))$. 

\[ \blacksquare \]
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Now we shall present a non-commutative extension of Theorem 2.

Theorem 7. A (CNT$(M_1)$) sequence $a_n : M \to L$ of additive maps is also (CLS$(M_1)$).

Proof. Let $\bar{x}$ belong to the $t_\tau$-closure of $C(M_1)$. By Proposition 1, $\bar{x} \in M_1$. Fix $\epsilon > 0$. Since, by Lemma 4, the sequence $\{a_n\}$ is (CNT$(M_2)$), for every $k \in \mathbb{N}$, there is $\gamma_k > 0$ such that, given $x \in M_2$ with $\rho(0,x) < \gamma_k$, one can find a projection $p_k = p_k(x) \in P(M)$, $\tau(p_k^k) \leq \epsilon/2^k$, satisfying $S(x,p_k) \leq 1/k$. Let a sequence $\{y_n\} \subset C(M_1)$ be such that $\rho(\bar{x},y_k) < \gamma_k$. If we set $x_k = y_k - \bar{x}$, then $x_k \in M_2$, $\rho(0,x_k) = \rho(\bar{x},x_k + \bar{x}) = \rho(\bar{x},y_k) \leq \gamma_k$, and $\bar{x} + x_k = y_k \in C(M_1)$, $k = 1, 2, \ldots$. If $e_k = p_k(x_k)$, then $\tau(e_k) \leq \epsilon/2^k$ and also $S(x_k,e_k) \leq 1/k$. Defining $e = \wedge_{k=1}^\infty e_k$, we obtain $\tau(e) \leq \epsilon$ and $S(x_k,e) \leq 1/k$. Therefore, by Lemma 2, the sequence $\{a_n(\bar{x})\}$ converges a.u., i.e. $\bar{x} \in C(M_1)$.

The following is an immediate consequence of the previous results of this section.

Theorem 8. Let $a_n : M \to L$ be a sequence of positive $t_\tau$-continuous linear maps such that $a_n(I) \leq I$, $n = 1, 2, \ldots$. If $\{a_n\}$ is (CNV$(D)$) with $D$ being $t_\tau$-dense in $M_1$, then conditions (CNV$(M_1)$), (CNT$(M_1)$), and (CLS$(M_1)$) are equivalent.

5 Conclusion

First we would like to stress that, due to Theorem 6, when establishing the almost uniform convergence of a sequence $\{a_n(x)\}$ for all $x \in L^\infty(M,\tau) = M$, the uniform equicontinuity at 0 on $M_1$ of the sequence $\{a_n\}$ is assumed. Also, as it is noticed in [1], the above formulation is important because, for example, if $\{a_n\}$ are bounded operators in a non-commutative $L^p$-space, $1 \leq p < \infty$, one may want to show that not only do these operators fail to converge a.u., but they fail so badly that $\{a_n\}$ may fail to converge a.u. on any class of operators which is $t_\tau$-dense in $M$.

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