Mathematical Analysis of a Generalized Chiral Quark Soliton Model

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Abstract. A generalized version of the so-called chiral quark soliton model (CQSM) in nuclear physics is introduced. The Hamiltonian of the generalized CQSM is given by a Dirac type operator with a mass term being an operator-valued function. Some mathematically rigorous results on the model are reported. The subjects included are: (i) supersymmetric structure; (ii) spectral properties; (iii) symmetry reduction; (iv) a unitarily equivalent model.

Key words: chiral quark soliton model; Dirac operator; supersymmetry; ground state; symmetry reduction

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1 Introduction

The chiral quark soliton model (CQSM) [5] is a model describing a low-energy effective theory of the quantum chromodynamics, which was developed in 1980’s (for physical aspects of the model, see, e.g., [5] and references therein). The Hamiltonian of the CQSM is given by a Dirac type operator with iso-spin, which differs from the usual Dirac type operator in that the mass term is a matrix-valued function with an effect of an interaction between quarks and the pion field. It is an interesting object from the purely operator-theoretical point of view too. But there are few mathematically rigorous analyses for such Dirac type operators (e.g., [2], where the problem on essential self-adjointness of a Dirac operator with a variable mass term given by a scalar function is discussed).

In the previous paper [1] we studied some fundamental aspects of the CQSM in a mathematically rigorous way. In this paper we present a slightly general form of the CQSM, which we call a generalized CQSM, and report that results similar to those in [1] hold on this model too, at least, as far as some general aspects are concerned.

2 A Generalized CQSM

The Hilbert space of a Dirac particle with mass $M > 0$ and iso-spin $1/2$ is taken to be $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$. For a generalization, we replace the iso-spin space $\mathbb{C}^2$ by an arbitrary complex Hilbert space $\mathcal{K}$. Thus the Hilbert space $\mathcal{H}$ in which we work in the present paper is given by

$$\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{K}.$$ 

We denote by $\mathcal{B}(\mathcal{K})$ the Banach space of all bounded linear operators on $\mathcal{K}$ with domain $\mathcal{K}$. Let $T : \mathbb{R}^3 \to \mathcal{B}(\mathcal{K}); \mathbb{R}^3 \ni \mathbf{x} = (x_1, x_2, x_3) \mapsto T(\mathbf{x}) \in \mathcal{B}(\mathcal{K})$ be a Borel measurable mapping
such that, for all \( x \in \mathbb{R}^3 \), \( T(x) \) is a non-zero bounded self-adjoint operator on \( K \) such that
\[
||T||_{\infty} := \sup_{x \in \mathbb{R}^3} ||T(x)|| < \infty,
\]
where \( ||T(x)|| \) denotes the operator norm of \( T(x) \).

**Example 1.** In the original CQSM, \( K = C^2 \) and \( T(x) = \tau \cdot n(x) \), where \( n : \mathbb{R}^3 \to \mathbb{R}^3 \) is a measurable vector field with \( |n(x)| = 1 \), a.e. (almost everywhere) \( x \in \mathbb{R}^3 \) and \( \tau = (\tau_1, \tau_2, \tau_3) \) is the set of the Pauli matrices.

We denote by \( \{\alpha_1, \alpha_2, \alpha_3, \beta\} \) the Dirac matrices, i.e., \( 4 \times 4 \)-Hermitian matrices satisfying
\[
\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad \{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j, k = 1, 2, 3,
\]
where \( \{A, B\} := AB + BA \).

Let \( U_F := (\cos F) \otimes I + i(\sin F)\gamma_5 \otimes T \),
where \( I \) denotes identity and \( \gamma_5 := -i\alpha_1\alpha_2\alpha_3 \). We set \( \alpha := (\alpha_1, \alpha_2, \alpha_3) \) and \( \nabla := (D_1, D_2, D_3) \) with \( D_j \) being the generalized partial differential operator in the variable \( x_j \). Then the one particle Hamiltonian of a generalized CQSM is defined by
\[
H := -i\alpha \cdot \nabla \otimes I + M(\beta \otimes I)U_F
\]
acting in the Hilbert space \( \mathcal{H} \). For a linear operator \( L \), we denote its domain by \( D(L) \). It is well-known that \( -i\alpha \cdot \nabla \) is self-adjoint with domain \( D(-i\alpha \cdot \nabla) = \cap_{j=1}^{3} D(D_j) \). Since the operator \( M(\beta \otimes I)U_F \) is bounded and self-adjoint, it follows that \( H \) is self-adjoint with domain \( D(H) = \cap_{j=1}^{3} D(D_j \otimes I) = H^1(\mathbb{R}^3; C^4 \otimes K) \), the Sobolev space of order 1 consisting of \( C^4 \otimes K \)-valued measurable functions on \( \mathbb{R}^3 \). In the context of the CQSM, the function \( F \) is called a profile function. In what follows we sometimes omit the symbol of tensor product \( \otimes \) in writing equations down.

**Example 2.** Usually profile functions are assumed to be rotation invariant with boundary conditions
\[
F(0) = -\pi, \quad \lim_{|x| \to \infty} F(x) = 0.
\]
The following are concrete examples [6]:

(I) \( F(x) = -\pi \exp(-|x|/R), \quad R = 0.55 \times 10^{-15} \) m;

(II) \( F(x) = -\pi \{a_1 \exp(-|x|/R_1) + a_2 \exp(-|x|^2/R_2^2)\}, \quad a_1 = 0.65, \quad R_1 = 0.58 \times 10^{-15} \) m, \( a_2 = 0.35, \quad R_2 = \sqrt{0.3} \times 10^{-15} \) m;

(III) \( F(x) = -\pi \left(1 - \frac{|x|}{\sqrt{\lambda^2 + |x|^2}}\right), \quad \lambda = \sqrt{0.4} \times 10^{-15} \) m.

We say that a self-adjoint operator \( \alpha \) on \( \mathcal{H} \) has chiral symmetry if \( \gamma_5 \alpha \subset \alpha \gamma_5 \).

**Proposition 1.** The Hamiltonian \( H \) has no chiral symmetry.

**Proof.** It is easy to check that, for all \( \psi \in D(H) \), \( \gamma_5 \psi \in D(H) \) and \( [\gamma_5, H] \psi = 2M\gamma_5 \beta U_F \psi \). Note that \( U_F \neq 0 \). Hence, \( [\gamma_5, H] \neq 0 \) on \( D(H) \).

We note that, if \( F \) and \( T \) are differentiable on \( \mathbb{R}^3 \) with \( \sup_{x \in \mathbb{R}^3} |\partial_j F(x)| < \infty \) and \( \sup_{x \in \mathbb{R}^3} |\partial_j T(x)| < \infty \) \( (j = 1, 2, 3) \), then the square of \( H \) takes the form
\[
H^2 = (-\Delta + M^2) \otimes I - iM\beta \alpha \cdot (\nabla U_F) + M^2 \sin^2 F \otimes (T^2 - I).
\]
This is a Schrödinger operator with an operator-valued potential.
3 Operator matrix representation

For more detailed analyses of the model, it is convenient to work with a suitable representation of the Dirac matrices. Here we take the following representation of $\alpha_j$ and $\beta$ (the Weyl representation):

$$
\alpha_j = \left( \begin{array}{cc} \sigma_j & 0 \\ 0 & -\sigma_j \end{array} \right), \quad \beta = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
$$

where $\sigma_1, \sigma_2$ and $\sigma_3$ are the Pauli matrices. Let $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ and

$$
\Phi_F := (\cos F) \otimes I + i(\sin F) \otimes T.
$$

Then we have the following operator matrix representation for $H$:

$$
H = \begin{pmatrix}
-i \sigma \cdot \nabla & M\Phi_F^* \\
M\Phi_F & i \sigma \cdot \nabla
\end{pmatrix}.
$$

4 Supersymmetric aspects

Let $\xi : \mathbb{R}^3 \rightarrow \mathcal{B}(\mathcal{K})$ be measurable such that, for all $x \in \mathbb{R}^3$, $\xi(x)$ is a bounded self-adjoint operator on $\mathcal{K}$ and $\xi(x)^2 = I, \forall x \in \mathbb{R}^3$. Let

$$
\Gamma(x) := i\gamma_5 \beta \otimes \xi(x), \quad x \in \mathbb{R}^3.
$$

We define an operator $\hat{\Gamma}$ on $\mathcal{H}$ by

$$(\hat{\Gamma}\psi)(x) := \Gamma(x)\psi(x), \quad \psi \in \mathcal{H}, \quad \text{a.e. } x \in \mathbb{R}^3.
$$

The following fact is easily proven:

**Lemma 1.** The operator $\hat{\Gamma}$ is self-adjoint and unitary, i.e., it is a grading operator on $\mathcal{H}$: $\hat{\Gamma}^* = \hat{\Gamma}$, $\hat{\Gamma}^2 = I$.

**Theorem 1.** Suppose that $\xi$ is strongly differentiable with $\sup_{x \in \mathbb{R}^3} \|\partial_j \xi(x)\| < \infty$ ($j = 1, 2, 3$) and

$$
\sum_{j=1}^3 \alpha_j \otimes D_j \xi(x) = M\gamma_5 \beta \{\xi(x), T(x)\} \sin F(x).
$$

Then $\hat{\Gamma}D(H) \subset D(H)$ and $\{\hat{\Gamma}, H\}\psi = 0, \forall \psi \in D(H)$.

**Proof.** For all $\psi \in D_0 := C_0^\infty(\mathbb{R}^3) \otimes_{\text{alg}} (\mathbb{C}^4 \otimes \mathcal{K})$ ($\otimes_{\text{alg}}$ denotes algebraic tensor product), we have

$$
D_j \hat{\Gamma}\psi = i\gamma_5 \beta \otimes (D_j \xi)\psi + i\gamma_5 \beta \otimes \xi(D_j\psi).
$$

By a limiting argument using the fact that $D_0$ is a core of $D_j \otimes I$, we can show that, for all $\psi \in D(D_j)$, $\hat{\Gamma}\psi$ is in $D(D_j)$ and (2) holds. Hence, for all $\psi \in D(H)$, $\hat{\Gamma}\psi \in D(H)$ and (2) holds. Thus we have for all $\psi \in D(H)$ \{\hat{\Gamma}, $H$\}$\psi = C_1\psi + C_2\psi$ with $C_1 := \sum_{j=1}^3 \{\gamma_5 \beta \otimes \xi, \alpha_j D_j\}$ and $C_2 := iM \{\gamma_5 \beta \otimes \xi, \beta U_F\}$. Using the fact that $\{\gamma_5, \beta\} = 0$ and $[\gamma_5, \alpha_j] = 0$ ($j = 1, 2, 3$), we obtain $C_1\psi = -\gamma_5 \beta (\sum_{j=1}^3 \alpha_j D_j \xi)\psi$. Similarly direct computations yield $(C_2\psi)(x) = -M \sin F(x) \otimes \{\xi(x), T(x)\}\psi(x)$. Thus (1) implies $\{\hat{\Gamma}, H\}\psi = 0$. ■
Theorem 1 means that, under its assumption, $H$ may be interpreted as a generator of a supersymmetry with respect to $\hat{\Gamma}$.

**Example 3.** Consider the case $\mathcal{K} = \mathbb{C}^2$. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable function such that
\[
(1 + C^2)f(x)^2 + g(x)^2 = 1.
\]
with a real constant $C \neq 0$ and $n(x) := (f(x), Cf(x), g(x))$. Then $|n(x)| = 1, \forall x \in \mathbb{R}^3$. Let
\[
\xi := \frac{C}{\sqrt{1 + C^2}}\tau_1 - \frac{1}{\sqrt{1 + C^2}}\tau_2, \quad T(x) := \tau \cdot n(x).
\]
Then $\xi^2 = I$ and $(\xi, T)$ satisfies (1).

To state spectral properties of $H$, we recall some definitions. For a self-adjoint operator $S$, we denote by $\sigma(S)$ the spectrum of $S$. The point spectrum of $S$, i.e., the set of all the eigenvalues of $S$ is denoted $\sigma_p(S)$. An isolated eigenvalue of $S$ with finite multiplicity is called a discrete eigenvalue of $S$. We denote by $\sigma_d(S)$ the set of all the discrete eigenvalues of $S$. The set $\sigma_{\text{ess}}(S) := \sigma(S) \setminus \sigma_d(S)$ is called the essential spectrum of $S$.

**Theorem 2.** Under the same assumption as in Theorem 1, the following holds:

(i) $\sigma(H)$ is symmetric with respect to the origin of $\mathbb{R}$, i.e., if $\lambda \in \sigma(H)$, then $-\lambda \in \sigma(H)$.

(ii) $\sigma_{\#}(H)$ ($\# = p, d$) is symmetric with respect to the origin of $\mathbb{R}$ with
\[
\dim \ker(H - \lambda) = \dim \ker(H - (-\lambda))
\]
for all $\lambda \in \sigma_{\#}(H)$.

(iii) $\sigma_{\text{ess}}(H)$ is symmetric with respect to the origin of $\mathbb{R}$.

**Proof.** Theorem 1 implies a unitary equivalence of $H$ and $-\hat{\Gamma}H\hat{\Gamma}^{-1} = -H$. Thus the desired results follow.

**Remark 1.** Suppose that the assumption of Theorem 1 holds. In view of supersymmetry breaking, it is interesting to compute $\dim \ker H$. This is related to the index problem: Let
\[
\mathcal{H}_+ := \ker(\hat{\Gamma} - 1), \quad \mathcal{H}_- := \ker(\hat{\Gamma} + 1)
\]
and
\[
H_{\pm} := H|\mathcal{H}_{\pm}.
\]
Then $H_+$ (resp. $H_-$) is a densely defined closed linear operator from $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) to $\mathcal{H}_-$ (resp. $\mathcal{H}_+$) with $D(H_+) = D(H)\cap \mathcal{H}_+$ (resp. $D(H_-) = D(H)\cap D(H_-)$). Obviously
\[
\ker H = \ker H_+ \oplus \ker H_-.
\]
The analytical index of $H_+$ is defined by
\[
\text{index}(H_+) := \dim \ker H_+ - \dim \ker H_+^*,
\]
provided that at least one of $\dim \ker H_+$ and $\dim \ker H_+^*$ is finite. We conjecture that, for a class of $F$ and $T$, $\text{index}(H_+) = 0$. 

5 The essential spectrum and finiteness of the discrete spectrum of $H$

5.1 Structure of the spectrum of $H$

**Theorem 3.** Suppose that $\dim K < \infty$ and

$$\lim_{|x| \to \infty} F(x) = 0. \tag{3}$$

Then

$$\sigma_{\text{ess}}(H) = (-\infty, -M] \cup [M, \infty), \tag{4}$$

$$\sigma_d(H) \subset (-M, M). \tag{5}$$

**Proof.** We can rewrite $H$ as $H = H_0 \otimes I + V$ with $H_0 := -i \alpha \cdot \nabla + M \beta$ and $V := M(\beta \otimes I)(U_F - I)$. We denote by $\chi_{R}(R > 0)$ the characteristic function of the set $\{x \in \mathbb{R}^3 | |x| < R\}$. It is well-known that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H_0 - z)^{-1}$ is compact [7, Lemma 4.6]. Since $K$ is finite dimensional, it follows that $(H_0 \otimes I - z)^{-1}$ is compact. We have

$$\|V(x)\| \leq M(\|F(x)\|_2 + \sin F(x)\|T\|_\infty).$$

Hence, by (3), we have $\lim_{R \to \infty} \sup_{|x| > R} \|V(x)\| = 0$. Then, in the same way as in the method described on [7, pp. 115–117], we can show that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H - z)^{-1}$ is compact. Hence, by a general theorem (e.g., [7, Theorem 4.5]), $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0 \otimes I)$. Since $\sigma_{\text{ess}}(H) = (-\infty, -M] \cup [M, \infty)$ ([7, Theorem 1.1]), we obtain (4). Relation (5) follows from (4) and $\sigma_d(H) = \sigma(H) \setminus \sigma_{\text{ess}}(H)$.

5.2 Bound for the number of discrete eigenvalues of $H$

Suppose that $\dim K < \infty$ and (3) holds. Then, by Theorem 3, we can define the number of discrete eigenvalues of $H$ counting multiplicities:

$$N_H := \dim \text{Ran} E_H((-M, M)), \tag{6}$$

where $E_H$ is the spectral measure of $H$.

To estimate an upper bound for $N_H$, we introduce a hypothesis for $F$ and $T$:

**Hypothesis (A).**

(i) $T(x)^2 = I$, $\forall x \in \mathbb{R}^3$ and $T$ is strongly differentiable with $\sum_{j=1}^{3} (D_j T(x))^2$ being a multiplication operator by a scalar function on $\mathbb{R}^3$.

(ii) $F \in C^1(\mathbb{R}^3)$.

(iii) $\sup_{x \in \mathbb{R}^3} |D_j F(x)| < \infty$, $\sup_{x \in \mathbb{R}^3} \|D_j T(x)\| < \infty$ ($j = 1, 2, 3$).

Under this assumption, we can define

$$V_F(x) := \sqrt{|\nabla F(x)|^2 + \sum_{j=1}^{3} (D_j T(x))^2 \sin^2 F(x)}. $$
Theorem 4. Let \( \dim K < \infty \). Assume (3) and Hypothesis (A). Suppose that

\[
C_F := \int_{\mathbb{R}^6} \frac{V_F(x)V_F(y)}{|x-y|^2} \, dxdy < \infty.
\]

Then \( N_H \) is finite with

\[
N_H \leq \frac{(\dim K)M^2C_F}{4\pi^2}.
\]

A basic idea for the proof of Theorem 4 is as follows. Let

\[
L(F) := H^2 - M^2.
\]

Then we have

\[
L(F) = -\Delta + M \begin{pmatrix}
0 & W_F^* \\
W_F & 0
\end{pmatrix}
\]

with \( W_F := i\sigma \cdot \nabla \Phi_F \). Note that

\[
W_F^*W_F = W_FW_F^* = V_F^2.
\]

Let

\[
L_0(F) := -\Delta - MV_F.
\]

For a self-adjoint operator \( S \), we introduce a set

\[
N_-(S) := \text{the number of negative eigenvalues of } S \text{ counting multiplicities}.
\]

The following is a key lemma:

Lemma 2.

\[
N_H \leq N_-(L(F)) \leq N_-(L_0(F)). \tag{7}
\]

Proof. For each \( \lambda \in \sigma_d(H) \cap (-M, M) \), we have \( \ker(H - \lambda) \subset \ker(L(F) - E_\lambda) \) with \( E_\lambda = \lambda^2 - M^2 < 0 \). Hence the first inequality of (7) follows. The second inequality of (7) can be proven in the same manner as in the proof of [1, Lemma 3.3], which uses the min-max principle. \( \square \)

On the other hand, one has

\[
N_-(L_0(F)) \leq \frac{(\dim K)M^2C_F}{4\pi^2}
\]

(the Birman–Schwinger bound [4, Theorem XIII.10]). In this way we can prove Theorem 4.

As a direct consequence of Theorem 4, we have the following fact on the absence of discrete eigenvalues of \( H \):

Corollary 1. Assume (3) and Hypothesis (A). Let \( (\dim K)M^2C_F < 4\pi^2 \). Then \( \sigma_d(H) = \emptyset \), i.e., \( H \) has no discrete eigenvalues.
6 Existence of discrete ground states

Let $A$ be a self-adjoint operator on a Hilbert space and bounded from below. Then

$$E_0(A) := \inf \sigma(A)$$

is finite. We say that $A$ has a ground state if $E_0(A) \in \sigma_p(A)$. In this case, a non-zero vector in $\ker(A - E_0(A))$ is called a ground state of $A$. Also we say that $A$ has a discrete ground state if $E_0(A) \in \sigma_d(A)$.

**Definition 1.** Let

$$E_0^+(H) := \inf [\sigma(H) \cap [0, \infty]], \quad E_0^-(H) := \sup [\sigma(H) \cap (-\infty, 0]].$$

(i) If $E_0^+(H)$ is an eigenvalue of $H$, then we say that $H$ has a positive energy ground state and we call a non-zero vector in $\ker(H - E_0^+(H))$ a positive energy ground state of $H$.

(ii) If $E_0^-(H)$ is an eigenvalue of $H$, then we say that $H$ has a negative energy ground state and we call a non-zero vector in $\ker(H - E_0^-(H))$ a negative energy ground state of $H$.

(iii) If $E_0^+(H)$ (resp. $E_0^-(H)$) is a discrete eigenvalue of $H$, then we say that $H$ has a discrete positive (resp. negative) energy ground state.

**Remark 2.** If the spectrum of $H$ is symmetric with respect to the origin of $\mathbb{R}$ as in Theorem 2, then $E_0^+(H) = -E_0^-(H)$, and $H$ has a positive energy ground state if and only if it has a negative energy ground state.

Assume Hypothesis (A). Then the operators

$$S_{\pm}(F) := -\Delta \pm M(D_3 \cos F)$$

are self-adjoint with $D(S_{\pm}(F)) = D(\Delta)$ and bounded from below.

As for existence of discrete ground states of the Dirac operator $H$, we have the following theorem:

**Theorem 5.** Let $\dim K < \infty$. Assume Hypothesis (A) and (3). Suppose that $E_0(S_+(F)) < 0$ or $E_0(S_-(F)) < 0$. Then $H$ has a discrete positive energy ground state or a discrete negative ground state.

**Proof.** We describe only an outline of proof. We have

$$\sigma_{\text{ess}}(L(F)) = [0, \infty), \quad \sigma_d(L(F)) \subset [-M^2, 0).$$

Hence, if $L(F)$ has a discrete eigenvalue, then $H$ has a discrete eigenvalue in $(-M, M)$. By the min-max principle, we need to find a unit vector $\Psi$ such that $\langle \Psi, L(F)\Psi \rangle < 0$. Indeed, for each $f \in D(\Delta)$, we can find vectors $\Psi_{f}^{\pm} \in D(L(F))$, such that $\langle \Psi_{f}^{\pm}, L(F)\Psi_{f}^{\pm} \rangle = \langle f, S_{\pm} f \rangle$. By the present assumption, there exists a non-zero vector $f_0 \in D(\Delta)$ such that $\langle f_0, S_+(F)f_0 \rangle < 0$ or $\langle f_0, S_-(F)f_0 \rangle < 0$. Thus the desired results follow.

To find a class of $F$ such that $E_0(S_+(F)) < 0$ or $E_0(S_-(F)) < 0$, we proceed as follows. For a constant $\varepsilon > 0$ and a function $f$ on $\mathbb{R}^d$, we define a function $f_\varepsilon$ on $\mathbb{R}^d$ by

$$f_\varepsilon(x) := f(\varepsilon x), \quad x \in \mathbb{R}^d.$$ 

The following are key Lemmas.
Lemma 3. Let $V : \mathbb{R}^d \to \mathbb{R}$ be in $L^2_{\text{loc}}(\mathbb{R}^d)$ and

$$S_\varepsilon := -\Delta + V_\varepsilon.$$ 

Suppose that:

(i) For all $\varepsilon > 0$, $S_\varepsilon$ is self-adjoint, bounded below and $\sigma_{\text{ess}}(S_\varepsilon) \subset [0, \infty)$.

(ii) There exists a nonempty open set $\Omega \subset \{ x \in \mathbb{R}^d | V(x) < 0 \}$.

Then there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $S_\varepsilon$ has a discrete ground state.

Proof. A basic idea for the proof of this lemma is to use the min-max principle (see \cite[Lemma 4.3]{1}). ■

Lemma 4. $V : \mathbb{R}^d \to \mathbb{R}$ be continuous with $V(x) \to 0(|x| \to \infty)$. Suppose that $\{ x \in \mathbb{R}^d | V(x) < 0 \} \neq \emptyset$. Then:

(i) $-\Delta + V$ is self-adjoint and bounded below.

(ii) $\sigma_{\text{ess}}(-\Delta + V) = [0, \infty)$.

(iii) $S_\varepsilon$ has a discrete ground state for all $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$.

Proof. The facts (i) and (ii) follow from the standard theory of Schrödinger operators. Part (iii) follow from a simple application of Lemma 3 (for more details, see the proof of \cite[Lemma 4.4]{1}). ■

We now consider a one-parameter family of Dirac operators:

$$H_\varepsilon := (-i\alpha \cdot \nabla + \frac{1}{\varepsilon}M(\beta \otimes I)U_{F_\varepsilon}).$$

Theorem 6. Let $\dim \mathcal{K} < \infty$. Assume Hypothesis (A) and (3). Suppose that $D_3 \cos F$ is not identically zero. Then there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $H_\varepsilon$ has a discrete positive energy ground state or a discrete negative ground state.

Proof. This follows from Theorem 5 and Lemma 4 (for more details, see the proof of \cite[Theorem 4.5]{1}). ■

7 Symmetry reduction of $H$

Let $T_1$, $T_2$ and $T_3$ be bounded self-adjoint operators on $\mathcal{K}$ satisfying

$$T_j^2 = I, \quad j = 1, 2, 3,$$

$$T_1T_2 = iT_3, \quad T_2T_3 = iT_1, \quad T_3T_1 = iT_2.$$ 

Then it is easy to see that the anticommutation relations

$$\{ T_j, T_k \} = 2\delta_{jk}I, \quad j, k = 1, 2, 3$$

hold. Since each $T_j$ is a unitary self-adjoint operator with $T_j \neq \pm I$, it follows that

$$\sigma(T_j) = \sigma_p(T_j) = \{ \pm 1 \}.$$ 

We set $\mathbf{T} = (T_1, T_2, T_3)$. 
In this section we consider the case where \( T(x) \) is of the following form:

\[
T(x) = n(x) \cdot T,
\]

where \( n(x) \) is the vector field in Example 1. We use the cylindrical coordinates for points \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \):

\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z,
\]

where \( \theta \in [0, 2\pi) \), \( r > 0 \). We assume the following:

**Hypothesis (B).** There exists a continuously differentiable function \( G : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) such that

(i) \( F(x) = G(r, z), \ x \in \mathbb{R}^3 \setminus \{0\} \);

(ii) \( \lim_{r + |z| \to \infty} G(r, z) = 0 \);

(iii) \( \sup_{r > 0, z \in \mathbb{R}} (|\partial G(r, z)/\partial r| + |\partial G(r, z)/\partial z|) < \infty \).

We take the vector field \( n : \mathbb{R}^3 \to \mathbb{R}^3 \) to be of the form

\[
n(x) := (\sin \Theta(r, z) \cos(m\theta), \sin \Theta(r, z) \sin(m\theta), \cos \Theta(r, z)),
\]

where \( \Theta : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) is continuous and \( m \) is a natural number.

Let \( L_3 \) be the third component of the angular momentum acting in \( L^2(\mathbb{R}^3) \) and

\[
K_3 := L_3 \otimes I + \frac{1}{2} \Sigma_3 \otimes I + \frac{m}{2} I \otimes T_3
\]

with \( \Sigma_3 := \sigma_3 \oplus \sigma_3 \). It is easy to see that \( K_3 \) is a self-adjoint operator acting in \( \mathcal{H} \).

**Lemma 5.** Assume that

\[
\Theta(\varepsilon r, \varepsilon z) = \Theta(r, z), \quad (r, z) \in (0, \infty) \times \mathbb{R}, \quad \varepsilon > 0.
\]

Then, for all \( t \in \mathbb{R} \) and \( \varepsilon > 0 \), the operator equality

\[
e^{itK_3} H_\varepsilon e^{-itK_3} = H_\varepsilon
\]

holds.

**Proof.** Similar to the proof of [1, Lemma 5.2]. We remark that, in the calculation of

\[
e^{itK_3}T(x)e^{-itK_3} = \sum_{j=1}^{3} e^{itL_3T_j(x)}e^{-itmT_3}e^{itmT_3},
\]

the following formulas are used:

\[
(T_1 \cos mt - T_2 \sin mt)e^{itmT_3} = T_1, \quad (T_1 \sin mt + T_2 \cos mt)e^{itmT_3} = T_2.
\]

**Definition 2.** We say that two self-adjoint operators on a Hilbert space strongly commute if their spectral measures commute.

**Lemma 6.** Assume (9). Then, for all \( \varepsilon > 0 \), \( H_\varepsilon \) and \( K_3 \) strongly commute.

**Proof.** By (10) and the functional calculus, we have for all \( s, t \in \mathbb{R} \)

\[
e^{itK_3}e^{isH_\varepsilon}e^{-itK_3} = e^{isH_\varepsilon},
\]

which is equivalent to \( e^{itK_3}e^{isH_\varepsilon} = e^{isH_\varepsilon}e^{itK_3}, \ s, t \in \mathbb{R} \). By a general theorem (e.g., [3, Theorem VIII.13]), this implies the strong commutativity of \( K_3 \) and \( H_\varepsilon \).
Lemma 6 implies that $H_{\varepsilon}$ is reduced by eigenspaces of $K_3$. Note that
\[
\sigma(K_3) = \sigma_p(K_3) = \left\{ \ell + \frac{s}{2} + \frac{mt}{2} \left| \ell \in \mathbb{Z}, s = \pm 1, t = \pm 1 \right. \right\}.
\]
The eigenspace of $K_3$ with eigenvalue $\ell + (s/2) + (mt/2)$ is given by
\[
\mathcal{M}_{\ell,s,t} := \mathcal{M}_\ell \otimes \mathcal{C}_s \otimes \mathcal{T}_t
\]
with $\mathcal{C}_s := \ker(\Sigma_3 - s)$ and $\mathcal{T}_t := \ker(T_3 - t)$. Then $\mathcal{H}$ has the orthogonal decomposition
\[
\mathcal{H} = \oplus_{\ell \in \mathbb{Z}, s,t \in \{\pm 1\}} \mathcal{M}_{\ell,s,t}.
\]
Thus we have:

**Lemma 7.** Assume (9). Then, for all $\varepsilon > 0$, $H_{\varepsilon}$ is reduced by each $\mathcal{M}_{\ell,s,t}$.

We denote by $H_{\varepsilon}(\ell, s, t)$ by the reduced part of $H_{\varepsilon}$ to $\mathcal{M}_{\ell,s,t}$ and set
\[
H(\ell, s, t) := H_1(\ell, s, t).
\]
For $s = \pm 1$ and $\ell \in \mathbb{Z}$, we define
\[
L_s(G, \ell) := -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \ell^2 + \frac{s^2}{r^2} + sMD_z \cos G
\]
acting in $L^2((0, \infty) \times \mathbb{R}, rdrdz)$ with domain
\[
D(L_s(G, \ell)) := C_0^\infty((0, \infty) \times \mathbb{R})
\]
and set
\[
\mathcal{E}_0(L_s(G, \ell)) := \inf_{f \in C_0^\infty((0, \infty) \times \mathbb{R}), \|f\|_{L^2((0, \infty) \times \mathbb{R}, rdrdz)} = 1} \langle f, L_s(G, \ell)f \rangle.
\]
The following theorem is concerned with the existence of discrete ground states of $H(\ell, s, t)$.

**Theorem 7.** Assume Hypothesis (B) and (9). Fix an $\ell \in \mathbb{Z}$ arbitrarily, $s = \pm 1$ and $t = \pm 1$. Suppose that $\dim T_\ell < \infty$ and
\[
\mathcal{E}_0(L_s(G, \ell)) < 0.
\]
Then $H(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.

**Proof.** Similar to the proof of Theorem 5 (for more details, see the proof of [1, Theorem 5.5]).

**Theorem 8.** Assume Hypothesis (B) and (9). Suppose that $\dim T_\ell < \infty$ and that $D_z \cos G$ is not identically zero. Then, for each $\ell \in \mathbb{Z}$, there exists a constant $\varepsilon_\ell > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\ell)$, each $H_{\varepsilon}(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.

**Proof.** Similar to the proof of Theorem 6 (for more details, see the proof of [1, Theorem 5.6]).

Theorem 8 immediately yields the following result:

**Corollary 2.** Assume Hypothesis (B) and (9). Suppose that $\dim T_\ell < \infty$ and that $D_z \cos G$ is not identically zero. Let $\varepsilon_\ell$ be as in Theorem 8 and, for each $n \in \mathbb{N}$ and $k > n$ $(k, n \in \mathbb{Z})$, $\nu_{k,n} := \min_{n+1 \leq \ell \leq k} \varepsilon_\ell$. Then, for each $\varepsilon \in (0, \nu_{k,n})$, $H_{\varepsilon}$ has at least $(k - n)$ discrete eigenvalues counting multiplicities.

**Proof.** Note that $\sigma_p(H_{\varepsilon}) = \cup_{\ell \in \mathbb{Z}, s,t = \pm 1} \sigma_p(H_{\varepsilon}(\ell, s, t))$. 

8 A unitary transformation

We go back again to the generalized CQSM defined in Section 2. It is easy to see that the operator
\[ X_F := \frac{1 + \gamma_5}{2} \exp \left( iF \otimes \frac{T}{2} \right) + \frac{1 - \gamma_5}{2} \exp \left( -iF \otimes \frac{T}{2} \right) \]
is unitary. Under Hypothesis (A), we can define the following operator-valued functions:
\[ B_j(x) := \frac{1}{2} D_j[F(x)T(x)], \quad x \in \mathbb{R}^3, \quad j = 1, 2, 3. \]

We set
\[ B := (B_1, B_2, B_3) \]
and introduce
\[ H(B) := (-i)\alpha \cdot \nabla + M\beta - \sigma \cdot B \]
acting in \( \mathcal{H} \). Since \( \sigma \cdot B \) is a bounded self-adjoint operator, \( H(B) \) is self-adjoint with \( D(H(B)) = \cap_{j=1}^3 D(D_j \otimes I) \).

**Proposition 2.** Assume Hypothesis (A) and that \( T(x) \) is independent of \( x \). Then
\[ X_F H X_F^{-1} = H(B). \]

**Proof.** Similar to the proof of [1, Proposition 6.1].

Using this proposition, we can prove the following theorem:

**Theorem 9.** Let \( \dim K < \infty \). Assume Hypothesis (A) and that \( T(x) \) is independent of \( x \). Suppose that
\[ \lim_{|x| \to \infty} |\nabla F(x)| = 0. \]

Then
\[ \sigma_{\text{ess}}(H) = (-\infty, -M] \cup [M, \infty). \tag{11} \]

**Proof.** By Proposition 2, we have \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H(B)) \). By the present assumption, \( B_j(x) = D_j F(x)T(0)/2 \). Hence
\[ \sup_{|x| > R} \|\sigma \cdot B(x)\| \leq \sum_{j=1}^3 \|T(0)/2\| \sup_{|x| > R} |D_j F(x)| \to 0 \quad (R \to \infty). \]

Therefore, as in the proof of Theorem 3, we conclude that \( \sigma_{\text{ess}}(H(B)) = (-\infty, -M] \cup [M, \infty] \). Thus (11) follows.
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