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Pachner Move 3 → 3 and Affine Volume-Preserving Geometry in $\mathbb{R}^3$

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Abstract. Pachner move 3 → 3 deals with triangulations of four-dimensional manifolds. We present an algebraic relation corresponding in a natural way to this move and based, a bit paradoxically, on *three*-dimensional geometry.

Key words: piecewise-linear topology; Pachner move; algebraic relation; three-dimensional affine geometry

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1 Introduction

Pachner moves are elementary rebuildings of a manifold triangulation. One can transform any triangulation of a fixed manifold into any other triangulation using a sequence of enough Pachner moves[1]. In any given dimension n of a manifold, there exist a finite number, namely $n+1$, types of Pachner moves. In dimension 4, with which we will deal here, these are moves $3 \to 3$, $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$, where the numbers show how many 4-simplices are taken away from the triangulation and with how many simplices they are replaced.

In general, if "elementary rebuildings" of some object are defined, then there is an interesting problem of finding algebraic formulas corresponding, in a sense, to these rebuildings. For Pachner moves, the immediate aim of such formulas may be the construction of manifold invariants based on them. It appears, however, that the real significance of such formulas is both wider and deeper. A good example of this is provided by the Onsager star-triangle relation and its generalization — Yang–Baxter equation. The latter proves to be relevant, in its various versions, to as different scientific areas as statistical physics, classical soliton equations, knot theory and the theory of left-symmetric algebras[2].

Algebraic relations between quantities of geometric origin, corresponding in a natural way to Pachner moves in dimensions 3 and 4, have been constructed in papers [3, 4, 5, 6]. These relations deserve the name of "geometric-semiclassical" because, in the case of three-dimensional Euclidean geometry [3, 5], they can be obtained by a semiclassical limit from the pentagon equation for $6j$-symbols, according to the Ponzano–Regge–Roberts formula [7, 8][3]. And when we are using other kinds of geometry, such as area-preserving plane geometry in [6, 10], or in the four-dimensional Euclidean case [4, 11, 12, 13], we think that our formulas still deserve to be called geometric-semiclassical: the constructions are always very clearly akin to those in the three-dimensional Euclidean case. The problem is, however, that the corresponding quantum formulas are not known! This problem seems to be hard (its solution will perhaps lead to new topological field theories in the spirit of Atiyah's axioms [14]), but it is worth some

[1]See the excellent paper [1] for exact theorems.
[2]One can consult the paper [2] about Yang–Baxter equation in the theory of left-symmetric algebras.
[3]See also similar formulas for spherical geometry in [9].

effort. To begin, one can search for as many as possible (semi)classical formulas, having in mind the following way of quantization known from the experience in studying integrable models in mathematical physics:

$$\begin{matrix}\text{classical}\\ \text{scalar model}\end{matrix} \xrightarrow{\text{generalization}} \begin{matrix}\text{classical}\\ \text{multicomponent}\\ \text{model}\end{matrix} \xrightarrow{\text{reduction}} \begin{matrix}\text{quantum}\\ \text{model}\end{matrix}.$$

Presenting here a "four-dimensional" relation, we mean not just its applications to topology of four-manifolds but also possible reduction(s) from four to three dimensions. In general, we have in mind the following way of constructing large families of manifold invariants: starting from a manifold M, we construct a simplicial complex K of a greater dimension — the reader can imagine a double cone over M just for a simple example — and consider a suitable invariant of K. We hope to demonstrate in one of future papers that this leads to interesting algebraic structures already for a two-dimensional M (even though we know everything about two-dimensional topology).

The direct aim of the present note is generalization of the "SL(2)-solution of pentagon equation" from [6, 10]. Here "solution of pentagon equation" means an algebraic relation corresponding in a natural sense to a move $2 \to 3$ of a three-dimensional manifold triangulation; instead of group SL(2), we now prefer to relate the constructions of [6, 10] to the group of affine area-preserving transformations of the plane $\mathbb{R}^2$. Below, we only generalize the pentagon relation of paper [6]; we plan to present the analogue of the acyclic complex from [10], which yields the actual manifold invariant, in a later publication.

2 The $3 \to 3$ relation: recalling the Euclidean case

The move $3 \to 3$ occupies, one can say, the central place among the Pachner moves in dimension four. Under this move, a cluster of three adjacent 4-simplices is replaced with a cluster of other three adjacent 4-simplices having the same boundary. We will construct an algebraic relation corresponding to this move, satisfied by some quantities λ related to tetrahedron volumes in a *three*-dimensional space $\mathbb{R}^3$. Besides these 3-volumes, we will be using linear dependencies among vectors joining points in $\mathbb{R}^3$ (e.g., in formula (4) below). Both volumes and linear dependencies are preserved under the action of the group of *volume-preserving affine transformations*, and in this sense our constructions are based on three-dimensional volume-preserving affine geometry.

An analogous algebraic relation, but using *four-dimensional Euclidean* geometry, was proposed in paper [11], and then in [12, 13] an invariant of 4-manifolds has been constructed based on this. As the analogy with Euclidean case was actually guiding us when dealing with other geometries, it makes sense to recall here, after some necessary explanations, the formula (8) from paper [11].

In considering the Pachner move $3 \to 3$, we will be dealing with six vertices in the triangulation, we call them A, B, C, D, E, and F. The sequence of letters $ABCDEF$ with, say, letter D taken away, will be denoted as $\hat{D}$. So, the Pachner move $3 \to 3$ starts with 4-simplices $ABCDE = \hat{F}$, $ABCFD = -\hat{E}$ and $ABCEF = \hat{D}$ (the minus sign is responsible for the orientation, and we choose a consistent orientation for 4-simplices) having the common two-dimensional face ABC, and replaces them with 4-simplices $\hat{A}$, $-\hat{B}$ and $\hat{C}$ having the common two-dimensional face DEF and the same common boundary.

Imagine now that points $A, \dots, F$ are placed in the space $\mathbb{R}^4$ *equipped with Euclidean metric.* This means, in particular, that the edges of 4-simplices acquire Euclidean lengths, the two-dimensional faces — areas (we denote, e.g., the area of ABC as S_{ABC}), and 4-simplices —

4-volumes (we denote the 4-volume of $\hat{A}$ as $V_{\hat{A}}$)[4]. The lengths of 15 edges joining the points $A, \dots, F$ are not independent — they satisfy the well-known relation: the *Cayley–Menger determinant* made of these lengths must vanish.

We will, however, represent this relation in a different form, namely in terms of *dihedral angles at a two-dimensional face*. It is clear that Euclidean simplices $\hat{F}$, $-\hat{E}$ and $\hat{D}$, if the lengths of their 15 edges are given, can be placed together in $\mathbb{R}^4$ if and only if the sum $\vartheta^{\hat{F}}_{ABC} + \vartheta^{-\hat{E}}_{ABC} + \vartheta^{\hat{D}}_{ABC}$ of dihedral angles at face ABC in these three 4-simplices is 2π (to be exact, their *algebraic* sum must be 0 modulo 2π: a dihedral angle must be taken sometimes with a minus sign — see [11, Section 3] for details). What will happen if we slightly, but arbitrarily, deform these 15 lengths? Each 4-simplex individually still can be put in $\mathbb{R}^4$, of course, but if one tries to put them all together, "cracks" will inevitably occur if the *deficit angle*

$$\omega_{ABC} \stackrel{\text{def}}{=} 2\pi - \vartheta^{\hat{F}}_{ABC} - \vartheta^{-\hat{E}}_{ABC} - \vartheta^{\hat{D}}_{ABC}$$

is nonzero.

Now we are ready to write down our "Euclidean" formula [11, (8)]:

$$V_{\hat{D}} V_{-\hat{E}} V_{\hat{F}} \frac{d\omega_{ABC}}{S_{ABC}} = V_{\hat{A}} V_{-\hat{B}} V_{\hat{C}} \frac{d\omega_{DEF}}{S_{ABC}}. \tag{1}$$

It means the following. First, vertices $A, \dots, F$ are put in $\mathbb{R}^4$ equipped with Euclidean metric, with the condition that all six 4-simplices with their vertices in $A, \dots, F$ be nondegenerate. This supplies us with edge lengths, two-dimensional areas, and four-dimensional volumes. Then we vary the lengths *infinitesimally* but otherwise *arbitrarily*. This yields an infinitesimal deficit angle $d\omega_{ABC}$, as well as a similar infinitesimal angle $d\omega_{DEF}$ at the face DEF around which the 4-simplices $\hat{A}$, $-\hat{B}$ and $\hat{C}$ are situated.

The proof of formula (1) can be found in [11, Section 2]. We now emphasize the following: all quantities entering the l.h.s. of (1) belong to the 4-simplices $\hat{D}$, $\hat{E}$ and $\hat{F}$[5], while all quantities entering the r.h.s. of (1) belong to the 4-simplices $\hat{A}$, $\hat{B}$ and $\hat{C}$. In this sense, (1) is a relation corresponding to the move $3 \to 3$.

The further way leading to invariants of four-dimensional PL manifolds goes through the use of acyclic complexes built from vector spaces consisting of differentials of geometric quantities [12, 13]. In the present paper, we do not consider these questions.

3 The $3 \to 3$ relation: affine volume-preserving geometry

Our work on invariants based on differential relations between geometric values attached to the elements of a manifold triangulation started not from 4-, but from 3-manifolds, and at first we were using only Euclidean geometry [3, 5]. The development of this subject proceeded, on the one hand, towards four dimensions, as described in the previous section, and on the other hand — towards using other (not Euclidean) geometries. For example, spherical geometry can be successfully used here, as demonstrated by Y. Taylor and C. Woodward [9]. It turned out also that affine volume-preserving geometry in $\mathbb{R}^{n-1}$ can be used, where n is the dimension of the manifold, which was shown in papers [6, 10] for $n = 3$. In this case, we were able to introduce some analogues of dihedral angles and deficit angles, so that the latter measured, so to say, the size of the cracks appearing when one tries to place a certain geometric configuration into $\mathbb{R}^{n-1} = \mathbb{R}^2$. This geometric configuration can be imagined as follows: take three tetrahedra $ABDE$, $BCDE$ and $CADE$ in $\mathbb{R}^3$ having a common edge DE, then project this picture onto $\mathbb{R}^2$.

[4] It is interesting to note that no *three*-volumes enter in the formula [11, (8)], presented below as (1). In contrast with this, our "affine volume-preserving" formula (12) is based exactly on three-volumes.

[5] When we are not preoccupied with being pedantic about the orientation, we drop the minus sign at $\hat{E}$ and $\hat{B}$.

Then, there is a quadratic relation between the areas of projections of two-dimensional faces. If we, however, allow these areas to be arbitrary, then our "deficit angles" measure to what degree this quadratic relation is violated. This works also for more than three tetrahedra situated in a similar way around a common edge.

In this light, the aim of the present paper looks natural — to construct an algebraic relation corresponding to a move $3 \to 3$ on a triangulation of a 4-manifold, and based on affine volume-preserving geometry in $\mathbb{R}^3$. As in the previous section, we will deal with vertices $A, \dots, F$ and 4-simplices $\hat{A}, \dots, \hat{F}$. As we take the geometry of $\mathbb{R}^3$ for the source of our algebraic relation, we keep in mind two pictures at once: the four-dimensional combinatorial-topological picture and the geometrical picture, where *three* coordinates are ascribed to each of vertices $A, \dots, F$ and one can thus speak of volumes of 3-simplices, i.e. tetrahedra. These volumes will play the key rôle below. We will allow ourselves to denote them by the same letter V as the 4-volumes in Section 2 (as we are not going to consider any 4-volumes from now on).

Although what follows is a generalization of the result of paper [6], using also some ideas from [10], our further exposition will be self-contained. Still, the analogy of our case $n = 4$ with the case $n = 3$ of those papers will be one more guiding thread for us.

There are $6!/(4!\,2!) = 15$ tetrahedra with vertices in points $A, \dots, F$. Their volumes are not independent. First, there are *linear* relations among them: the overall volume of the boundary of any 4-simplex is zero, e.g.,

$$V_{BCDE} - V_{ACDE} + V_{ABDE} - V_{ABCE} + V_{ABCD} = 0. \tag{2}$$

Certainly, we are always speaking of oriented volumes: $V_{ABCD} = -V_{BACD}$, etc.

There are five independent relations of type (2). Besides, there must exist three more relations, which can be seen from the following counting of parameters (which is no problem to be made rigorous): 6 points $A, \dots, F$ have 18 coordinates. From these, one must subtract the number of parameters in the group of affine volume-preserving motions of $\mathbb{R}^3$, namely 8 parameters corresponding to the subgroup $\mathrm{SL}(3, \mathbb{R})$ and 3 translations. Now everything is all right, the number of volumes minus the number of relations imposed on them is equal to the number of coordinates minus the numbers of motions:

$$15 - 5 - 3 = 18 - 8 - 3. \tag{3}$$

We will obtain the three additional (nonlinear) relations among volumes in such way which will lead us to the natural analogues of dihedral and deficit angles for our non-metric geometry. Consider the cluster of three initial 4-simplices $\hat{F}$, $-\hat{E}$, $\hat{D}$ grouped around their common two-dimensional face ABC. First, we write down the following easily derived relation which expresses the vector $\overrightarrow{CE}$ (of course, it lies in $\mathbb{R}^3$) belonging to the 3-face $ABCE$ of 4-simplex $\hat{F}$ in terms of the vector $\overrightarrow{CD}$ belonging to the 3-face $ABCD$ of $\hat{F}$, the 3-volumes of faces of $\hat{F}$, and two vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ belonging to ABC:

$$\overrightarrow{CE} = \frac{-V_{BCDE}\overrightarrow{CA} + V_{ACDE}\overrightarrow{CB} + V_{ABCE}\overrightarrow{CD}}{V_{ABCD}}. \tag{4}$$

We say briefly that we have expressed $\overrightarrow{CE}$ through $\overrightarrow{CD}$ from 4-simplex $\hat{F}$. Similarly, we express $\overrightarrow{CF}$ through $\overrightarrow{CE}$ from $\hat{D}$ and $\overrightarrow{CD}$ through $\overrightarrow{CF}$ from $\hat{E}$. Taking the composition of these expressions, we can say that we go around the face ABC and thus express $\overrightarrow{CD}$ through itself. To be exact, the "new" $\overrightarrow{CD}$ comes out as a linear combination of the "old" $\overrightarrow{CD}$ and "fixed" vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$:

$$\overrightarrow{CD}_{\text{new}} = \overrightarrow{CD}_{\text{old}} + \omega_{ABC}^{(1)} V_{ABCD}\overrightarrow{CA} + \omega_{ABC}^{(2)} V_{ABCD}\overrightarrow{CB},$$

where

$$\omega_{ABC}^{(1)} = -\frac{V_{BCDE}}{V_{ABCD}V_{ABCE}} - \frac{V_{BCEF}}{V_{ABCE}V_{ABCF}} + \frac{V_{BCDF}}{V_{ABCF}V_{ABCD}} \tag{5}$$

and

$$\omega_{ABC}^{(2)} = \frac{V_{ACDE}}{V_{ABCD}V_{ABCE}} + \frac{V_{ACEF}}{V_{ABCE}V_{ABCF}} - \frac{V_{ACDF}}{V_{ABCF}V_{ABCD}}. \tag{6}$$

The two nonlinear relations

$$\omega_{ABC}^{(1)} = \omega_{ABC}^{(2)} = 0 \tag{7}$$

obviously guarantee that the cluster of $\hat{F}$, $\hat{D}$ and $\hat{E}$, with given 3-volumes of their 3-faces (satisfying, of course, the conditions of type (2)), can be placed in $\mathbb{R}^3$ without contradictions. With nonzero ω's, this cannot be done, although each 4-simplex individually still can be put in $\mathbb{R}^3$. We note that each of the quantities $\omega_{ABC}^{(1)}$ and $\omega_{ABC}^{(2)}$ is a sum of three expressions of the same kind made of values belonging to 4-simplices $\hat{F}, \hat{D}, \hat{E}$ respectively. We can call them components of two-component "dihedral angles" at the face ABC in the mentioned simplices: for instance, the "dihedral angle" in $\hat{F}$ is $\Big(-V_{BCDE}/(V_{ABCD}V_{ABCE}),\ V_{ACDE}/(V_{ABCD}V_{ABCE})\Big)$. The quantities $\omega_{ABC}^{(1)}$ and $\omega_{ABC}^{(2)}$ can be called components of the "deficit angle", or "discrete curvature" around ABC.

There must also be a third nonlinear relation between three-dimensional volumes. As we have shown that it is not necessary for putting in $\mathbb{R}^3$ the 4-simplices $\hat{D}$, $\hat{E}$ and $\hat{F}$, it must involve the volumes of 3-faces absent from these 4-simplices. So, now we consider in the same way 4-simplices $\hat{A}$, $\hat{B}$ and $\hat{C}$, of which we think as grouped around the face DEF. We can just change the letters in formulas (5) and (6) as follows: $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$, which means exactly coming from the initial 4-simplices $\hat{F}$, $\hat{D}$ and $\hat{E}$ to the simplices $\hat{C}$, $\hat{A}$ and $\hat{B}$ obtained as a result of the move $3 \to 3$. If we now write out the conditions

$$\omega_{DEF}^{(1)} = \omega_{DEF}^{(2)} = 0 \tag{8}$$

obtained from (7) under this change, we can check (e.g., on a computer) that (7) and (8) together give exactly three independent conditions.

We now express the three-dimensional volumes through values λ attached to two-dimensional faces, according to the following pattern:

$$V_{ABCD} = \lambda_{BCD} - \lambda_{ACD} + \lambda_{ABD} - \lambda_{ABC}. \tag{9}$$

In this way we, of course, guarantee that the five linear relations among volumes hold automatically. In doing so, we act in analogy with papers [6, 10], where we had similar quantities λ attached to *edges* of triangulation. Although we do not permute the subscripts of λ in this paper, it is natural to consider λ as totally antisymmetric in its indices.

Now we can guess how the algebraic relation corresponding to move $3 \to 3$ can look. In the left-hand side, which corresponds to simplices $\hat{D}$, $\hat{E}$ and $\hat{F}$, we expect to find quantities belonging to the faces specific for that set of simplices: 2-face ABC and 3-faces $ABCD$, $ABCE$ and $ABCF$, while in the right-hand side we expect quantities belonging to the 2-face DEF and 3-faces $ADEF$, $BDEF$ and $CDEF$. There is no problem about 3-faces: natural quantities belonging to them are their volumes. As for the 2-faces, to each of them belong two components of ω and one λ, and, as papers [6, 10] suggest, we must make something like a derivative $\partial\omega/\partial\lambda$. The right answer (justified finally by a computer) is to take, for the face ABC, a Jacobi determinant $J_{ABC,i}$ of partial derivatives of $\omega_{ABC}^{(1)}$ and $\omega_{ABC}^{(2)}$ with respect to λ_{ABC} and one more λ, call it λ_i, where i denotes a two-dimensional face coinciding with neither ABC nor DEF:

$$J_{ABC,i} = \begin{vmatrix} \partial\omega^{(1)}_{ABC}/\partial\lambda_{ABC} & \partial\omega^{(1)}_{ABC}/\partial\lambda_i \\ \partial\omega^{(2)}_{ABC}/\partial\lambda_{ABC} & \partial\omega^{(2)}_{ABC}/\partial\lambda_i \end{vmatrix}. \tag{10}$$

All derivatives are taken in the "flat" point where all ω's are zero and thus all picture is genuinely in $\mathbb{R}^3$.

If we compose also the Jacobi determinant $J_{DEF,i}$ of partial derivatives of $\omega^{(1)}_{DEF}$ and $\omega^{(2)}_{DEF}$ with respect to λ_{DEF} and the same λ_i, then it is not hard to see that the ratio $J_{ABC,i}/J_{DEF,i}$ does not in fact depend on the face i. Indeed, if j is another face $\neq ABC, DEF$, then, according to a version of the theorem on implicit function derivative,

$$\frac{J_{ABC,i}}{J_{ABC,j}} = -\frac{\partial\lambda_j}{\partial\lambda_i}, \tag{11}$$

where the partial derivative in the right-hand side is taken under the condition that all ω's are zero. This condition means, as we explained already, that values λ are such that the points $A, \ldots, F$ can be placed in $\mathbb{R}^3$ in such way that the tetrahedron volumes coincide with values obtained from formulas of type (9); and in (11) the three quantities λ_j, λ_{ABC} and λ_{DEF} are regarded as functions of other λ's, including λ_i. It is clear, on the other hand, that $J_{DEF,i}/J_{DEF,j}$ equals the same right-hand side of (11), so, indeed, the ratio $J_{ABC,i}/J_{DEF,i}$ is independent of i.

What remains now is to calculate this ratio on a computer and verify our conjecture that it is expressed in a nice way through three-dimensional volumes of faces in which our clusters $\hat{D}$, $\hat{E}$, $\hat{F}$ and $\hat{A}$, $\hat{B}$, $\hat{C}$ differ from one another. It turns out that these volumes appear in the formula raised in the second power. Here is this formula — our formula corresponding to the move $3 \to 3$, proved by a symbolic calculation using Maple:

$$(V_{ABCD}V_{ABCE}V_{ABCF})^2 J_{ABC,i} = (V_{ADEF}V_{BDEF}V_{CDEF})^2 J_{DEF,i}. \tag{12}$$

4 Concluding remarks

The result of this note — formula (12) — relies upon computer-assisted calculations which were being made in the belief that a four-dimensional analogue of formula (15) from [6] must exist. Thus, a careful investigation of algebraic structures arising here is still to be done.

We emphasize once more that we consider the constructing of various new relations, such as (12), and investigating the related algebraic structures as a way to discovering new quantum topological field theories, in particular, for manifolds of higher (> 3) dimensions.

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