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ON THE NUMBER OF CONTROL SETS ON COMPACT HOMOGENEOUS SPACES

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Abstract: This paper gives conditions to determine the number of control sets on compact homogeneous spaces of a Lie group G. We use a Levi decomposition of G to reduce the problem to homogeneous spaces of semi-simple Lie groups and, in particular, to generalized flag manifolds.

1 – Introduction

One of the principal concepts in the study of control systems is the study of the controllability. Many questions related to the controllability depend, in fact, of the semigroup of transformations defined by the flow of the control system, known as the system's semigroup. Thus, the theory of controllability of control systems can be abstracted to arbitrary semigroup actions and solved in a more general setting. The regions of the state space of the control system where the controllability occurs are called control sets. The control sets for control systems were mainly studied by Colonius and Kliemann in [4], [5] and [6]. The generalization of the concept of a control set for semigroup actions was given by San Martin and Tonelli (see [11] and [12]). One important branch of investigation is the determination of the number of control sets. The study of the number of control sets on generalized flag manifolds (a compact homogeneous space which is the

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quotient of a semi-simple Lie group by a parabolic subgroup) was exploited by Braga Barros and San Martin (see [2], [3], [11] and [12]). In particular, San Martin [12] had shown that there is only one invariant control set in a flag manifold. The present article follows this line of questioning. It investigates the number of control sets on compact homogeneous spaces of Lie groups.

Let G be a connected and simple-connected Lie group. Suppose that L is a closed subgroup of G. A Levi decomposition of G is given by $G = RH$ where R is the radical of G and H is semi-simple. The main result of the paper says that, under certain conditions, the number of control sets on the homogeneous spaces G/L and $H/(H \cap RL)$ is the same. In particular, when $H \cap RL$ is a parabolic subgroup of the semi-simple Lie group H one has that $H/(H \cap RL)$ is a flag manifold and the number of control sets on flag manifolds were determined in [11] (see Proposition 5 bellow). Now, we give the idea of the proof of this result. First, we construct an associated fiber bundle (see definition bellow) with projection $G/L \to H/(H \cap RL)$ and typical fiber $R/(R \cap L)$. Since R is solvable it is shown that any semigroup with non-empty interior and generating G acts transitively on the homogeneous space $R/(R \cap L)$. To conclude the proof we use a result of [1] which says that the transitivity of the semigroup on the fibers over a control set implies that the number of control sets in the base space and in the total space is equal.

2 – Control sets and fiber bundles

In this section we recall the concepts of control sets and fiber bundles. We also relate these two concepts and present the main results that will be used later in the text.

First, we recall the concept of control sets for semigroup actions (see [1], [11] and [12] for results on the control sets for semigroup actions). Denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of the manifold M. We say that $S \subset \mathrm{Diff}(M)$ is a semigroup in case S is closed under compositions. For a set $S \subset \mathrm{Diff}(M)$ and $x \in M$, we use the notation

$$Sx = \{\phi(x) \colon \phi \in S\} .$$

for the orbit of x under S. From now on, and in the whole paper we assume that S and S^{-1} are semigroups satisfying the accessibility property, that is, $\mathrm{int}(Sx) \neq \emptyset$ and $\mathrm{int}(S^{-1}x) \neq \emptyset$ for every $x \in M$. A control set for S on M is a subset $D \subset M$ which satisfies

1. $\mathrm{int}(D) \neq \emptyset$,

2. $\forall x \in D$, $D \subset \mathrm{cl}(Sx)$ and

3. D is maximal with these properties.

As stated above, the control sets are the subsets where the semigroup is approximately transitive. This approximate transitivity can be improved to exact transitivity inside a dense subset of D. Let

$$D_0 = \left\{ x \in D\colon\ x \in \mathrm{int}(Sx) \cap \mathrm{int}(S^{-1}x) \right\} .$$

be the *set of transitivity* of the control set D. In general, D_0 may be empty. However, in case it is not empty, it is open and dense in D. One also has that for all $x, y \in D_0$ there exists $g \in S$ such that $gx = y$ (see [1] Proposition 2.2). The control set D is an *effective* control set in case $D_0 \neq \emptyset$. We also recall that a control set is called an *invariant* control set if it is invariant under the action of the semigroup S.

In order to continue our discussion we relate the control sets with fiber bundles. Therefore, it is convenient to recall the concepts of principal bundles and their associated bundles. We refer [8] and [9] for the theory of fiber bundles with details. Let G be a Lie group acting on the right and effectively on a manifold Q. We denote by

$$\begin{array}{rcl} Q \times G & \to & Q \\ (q, g) & \mapsto & q.g \end{array}$$

the right action of G on Q. We define

$$Q^* = \left\{ (q, q.g) \in Q \times Q\colon\ q \in Q \text{ and } g \in G \right\} .$$

Then for every $(q, q') \in Q^*$ there exists a function $\tau\colon Q^* \to G$ such that $q\,\tau(q, q') = q'$. The function τ is called the translation function and satisfies the following properties:

1. $\tau(q, q) = e$, here e is the identity of the group;

2. $\tau(q, q')\,\tau(q', q'') = \tau(q, q'')$ for every $q, q', q'' \in Q$,;

3. $\tau(q', q) = \tau(q, q')^{-1}$ for every $q, q' \in Q$.

A right action of G on Q is called *principal* if it is effective and has a differentiable translation $\tau\colon Q^* \to G$.

Let Q and M be manifolds. Suppose that the Lie group G acts on the right on Q and assume that the action is principal. We recall that a *principal* bundle is a quadruplet (Q, π_Q, M, G) where $\pi_Q\colon Q \to M$ is differentiable, open and surjective. The space M is the *base space*, the space Q is the *total space*, G is the *structure group* and π_Q is the *projection*.

Let (Q, π_Q, M, G) be a principal bundle and take $x \in M$. It is known that the fiber $\pi_Q^{-1}(x)$ is diffeomorphic to the structure group G and the diffeomorphism is given by the bijection $u\colon G \to \pi_Q^{-1}(x)$ defined by $u(g) = q.g$. Therefore, $\pi_Q^{-1}(x) = Gq = \{gq\colon g \in G\}$ with $q \in \pi_Q^{-1}(x)$ and the group G acts transitively on the fiber. Now, we introduce the concept of a bundle associated to the principal bundle.

Let (Q, π_Q, M, G) be a principal bundle and assume that the Lie group G acts on the left and transitively on a manifold F. Then G acts on the right on $Q\times F$ in the following way: for $(q, v) \in Q\times F$ and $g \in G$ we define $(q, v)g = (qg, g^{-1}v)$. Now, we consider a equivalence relation on $Q\times F$ defined by

$$(q_1, v_1) \sim (q_2, v_2) \quad \text{if and only if} \quad \exists\, g \in G \;\text{ such that } q_2 = q_1 g \;\text{ and }\; v_2 = g^{-1}v_1 \ .$$

Let E be the quotient space of $Q\times F$ by the relation $\sim$. We denote an element of E as qv. We also define the projection $\pi_E\colon E \to M$ by $\pi_E(qv) = \pi_Q(q)$, where $\pi_Q\colon Q \to M$ is the projection in the principal bundle. The fiber bundle with total space E, base space M and projection π_E is denoted by (E, π_E, M, F, G), and it is called the *bundle associated* to the principal bundle (Q, π_Q, M, G). We also say that G is the *structure group* of the associated bundle. The manifold F is called the *typical fiber* of the associated bundle.

Take $q_0 \in Q$ such that $\pi_Q(q_0) = x$ and let $f\colon F \to \pi_E^{-1}(x)$ be the bijection defined by $f(v) = q_0 v$. It follows that f is a diffeomorphism. Therefore the fiber of the associated bundle is diffeomorphic to F.

Let S_Q be a semigroup of diffeomorphisms of Q commuting with the right action, that is, $\phi(q.a) = \phi(q)a$, for every $a \in G$ if $\phi \in S_Q$. Then $\phi \in S_Q$ induces the diffeomorphism of E defined by $E\phi(qv) = \phi(q)v$. The semigroup S_Q of diffeomorphisms of Q induces a semigroup S_E of diffeomorphisms of E.

Given $q \in Q$ we define the subset

$$S_q = S_Q(q) \cap \pi_Q^{-1}(x)\,, \qquad x = \pi_Q(q)\ .$$

Through the identification of the fiber over x with G via $a \in G \longmapsto q.a \in \pi_Q^{-1}(x)$, S_q can be viewed as a subset of G

$$S_q = \Big\{a \in G\colon \exists\, \phi \in S_Q,\ \phi(q) = q.a\Big\}\ .$$

It follows that S_q is a subsemigroup of G if $S_q \neq \emptyset$. We observe that $a \in S_q$ acts on the typical fiber as $v.a = qv.a$.

In the following, we give examples of the concepts defined above where the total space and the base space of the bundles are homogeneous spaces of a Lie group G. We recall the concept of an equivariant fibration between homogeneous spaces. Let G/L_1 and G/L_2 be homogeneous spaces. A fibration $\pi\colon G/L_1 \to G/L_2$ is called *equivariant* if $\pi(xg) = \pi(x)g$ for every $x \in G/L_1$ and $g \in G$. Now, we go to the examples.

Example 1. Let G be a Lie group and $L_1 \subset L_2$ closed subgroups with L_1 normal in L_2. Consider the map

$$\begin{array}{rlcl} \pi\colon & G/L_1 & \to & G/L_2 \\ & gL_1 & \mapsto & gL_2 \end{array}$$

We have that $(G/L_1, \pi, G/L_2, L_2/L_1)$ defines a principal bundle. The right action of L_2/L_1 on G/L_1 is defined by

$$(gL_1)(hL_1) = ghL_1\,, \qquad g \in G, \quad h \in L_2$$

and commutes with the right action of G on G/L_1. One has $\pi(hL_1g) = \pi(hL_1)g$ and π is equivariant. If S is a semigroup in G and with $\mathrm{int}_G(S) \neq \emptyset$ then S induces a semigroup S_Q of diffeomorphisms of G/L_1

$$S_Q = \{\phi_s\colon\ s \in S\} \qquad \text{and} \qquad \phi_s(gL_1) = gsL_1\ .$$

In this case the semigroup S_M is the semigroup of diffeomorphisms of G/L_2 induced by S via the action of G on G/L_2. One also has that $S_q = (gS \cap L_2)/L_1$ if $q = gL_1 \in G/L_1$. In particular, $S_q = (S \cap L_2)/L_1$ if q is the coset L_1. □

Example 2. Let G be a Lie group and $L_1 \subset L_2$ closed subgroups. Define the map

$$\begin{array}{rlcl} \pi\colon & G & \to & G/L_2 \\ & g & \mapsto & gL_2 \end{array}$$

then $(G, \pi, G/L_2, L_2)$ defines a principal bundle. The right action of L_2 on G is defined by

$$gh\,, \qquad g \in G, \quad h \in L_2\ .$$

We also define

$$\begin{array}{rlcl} \pi_E\colon & G/L_1 & \to & G/L_2 \\ & gL_1 & \mapsto & gL_2 \end{array}$$

then $(G/L_1, \pi_E, G/L_2, L_2/L_1, L_2)$ is a fiber bundle associated to $(G, \pi, G/L_2, L_2)$. If S is a semigroup in G with $\mathrm{int}_G(S) \neq \emptyset$ then S induces a semigroup S_Q of diffeomorphisms of G

$$S_Q = \{\phi_s\colon\ s \in S\} \quad \text{and} \quad \phi_s(g) = gs \ .$$

In this case the semigroup S_M is the semigroup of diffeomorphisms of G/L_2 induced by S via the action of G on G/L_2. One also has that S induces a diffeomorphism S_E on G/L_1 defined

$$S_E = \{E\phi_s\colon\ s \in S\} \quad \text{and} \quad E\phi_s(gL_1) = gsL_1 \ .$$

We have that $S_q = gS \cap L_2$ if $q = g \in G$. In particular, $S_q = S \cap L_2$ if $q = 1 \in G$. □

In the discussion on the number of control sets, a stronger version of accessibility is needed (see Definition 3.1 in [1]). Let D be an effective control set for S_M. We recall that the semigroup S_Q is said to be accessible over D if for some, and hence for all, $q \in \pi_Q^{-1}(D_0)$, $\mathrm{int}(S_Q q) \cap \pi_Q^{-1}(D_0) \neq \emptyset$. Similarly, S_E is said to be accessible over D if $\mathrm{int}(S_E u) \cap \pi_E^{-1}(D_0) \neq \emptyset$. Now, we show, for equivariant fibrations, that the accessibility property defined above holds over any effective control set.

Proposition 1. *Let S be a subsemigroup of G with non-empty interior. Assume that G/L_1 and G/L_2 are homogeneous spaces of G and $\pi\colon G/L_1 \to G/L_2$ is an equivariant fibration. Suppose S induces semigroups S_E and S_M on $E = G/L_1$ and $M = G/L_2$, respectively. Then S_E is accessible over any effective control set for S_M.*

Proof: Let D be an effective control set for S_M on G/L_2. Take $gL_2 \in D_0$. By the definition of D_0 we have that there exists $s \in \mathrm{int}(S)$ such that $M\phi_s(gL_2) = (gL_2)s = gL_2$. Define $\phi(hL_1) = hsL_1$, $h \in G$. Thus $\phi \in S_E$. Since π is equivariant one has that $\pi(\phi(gL_1)) = gsL_2 = gL_2$ and $\phi(gL_1) \in \pi^{-1}(D_0)$. It is enough to show that $\phi(gL_1) \in \mathrm{int}(S_E(gL_1))$. This fact is true since $\phi(gL_1) = gsL_1 = gL_1 s \in gL_1 \mathrm{int}(S) \subset S_E(gL_1)$. Therefore $\phi(gL_1) \in \mathrm{int}(S_E(gL_1))$. ■

In the following we present the results on the number of control sets (and invariant control sets) that are used in the text. This results were shown in [1]. We start with a result on the number of invariant control sets.

Proposition 2. *Let $C \subset M$ be an invariant control set for S_M, and assume that S_Q is accessible over C. Assume also that the fiber F is compact. Then the number of invariant control sets for S_q in F with $q \in \pi_E^{-1}(C_0)$ is the same as the number of invariant control sets for S_E on E.*

Proof: Theorem 4.4 in [1]. ∎

One also has.

Proposition 3. *Assume that S_M is accessible. Suppose that for every effective control set $C \subset M$ there exists $x \in C_0$ and $q \in \pi_Q^{-1}(x)$ such that S_q acts transitively on F. Then $\pi_E^{-1}(C)$ is an effective control set on E if C is an effective control set in M. Therefore the number of effective control sets on E and on M is the same. Moreover, the same result is also true if we consider invariant control sets instead of effective control sets.*

Proof: It is Proposition 3.7 in [1] and the fact that invariant control sets projects into invariant control sets. ∎

A consequence of the last proposition is the corollary bellow.

Corollary 1. *Let (Q, π_Q, M, G) be a principal bundle such that its structure group G is compact and connected. Assume that S_Q is accessible over any control set on the base space M. Then $\pi_Q^{-1}(C)$ is an effective control set for S_Q on Q if C is an effective control set in M. Therefore the number of effective control sets on Q is the same as on M. Moreover, the same result is also true if we consider invariant control sets instead of effective control sets.*

Proof: Corollary 3.8 in [1]. ∎

Remark. In the last corollary we may assume that $\pi^{-1}(C_0)$ is connected instead of assuming that G is connected. □

As an immediate application of the corollary we have.

Proposition 4. *Let $L_1 \subset L_2$ be closed subgroups of the connected Lie group G with L_1 normal in L_2. Assume that G/L_1 is compact and that L_2/L_1 is connected. Suppose that the number of control sets on G/L_2 is finite. Then the number of effective control sets on G/L_1 is the same as on G/L_2.*

Proof: The fact that G/L_1 is compact implies that G/L_2 and L_2/L_1 are compact. Thus the projection $G/L_1 \to G/L_2$ determines a principal bundle with compact and connected structure group L_2/L_1. To finish the proof we apply Corollary 1. ∎

3 – Flag manifolds

We will be interested on flag manifolds, that is, homogeneous spaces G/H with G a non-compact semi-simple Lie group and H a parabolic subgroup. We refer the reader to [16] and [15] for the detailed theory of parabolic subgroups and flag manifolds. We use the following standard notation and terminology. Let $\mathfrak{g}$ be the Lie algebra of G. Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ with $\mathfrak{k}$ the compactly embedded subalgebra. Let $\mathfrak{a}$ be a maximal abelian subalgebra contained in $\mathfrak{s}$ and denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Fix a simple system of roots $\Sigma \subset \Pi$. Denote by Π^+ the set of positive roots and by $\mathfrak{a}^+$ the Weyl chamber

$$\mathfrak{a}^+ = \Big\{\mathfrak{H} \in \mathfrak{a} \colon \ \alpha(\mathfrak{H}) > \mathfrak{o} \ \text{ for all } \alpha \in \mathfrak{t}\Big\} .$$

Let

$$\mathfrak{n} = \sum_{\alpha \in \mathfrak{t}^+} \mathfrak{g}_\alpha$$

be the direct sum of the root spaces corresponding to the positive roots. The notations K, N are used to indicate the connected subgroups whose Lie algebras are $\mathfrak{k}$ and $\mathfrak{n}$ respectively. Let W be the Weyl group of G. It is constructed either as the subgroup of reflections generated by the roots of $(\mathfrak{g}, \mathfrak{a})$ or as the quotient $W = M^*/M$ where M^* and M are respectively the normalizer and the centralizer of $\mathfrak{a}$ in K. A minimal parabolic subalgebra of $\mathfrak{g}$ is given by

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} .$$

Let P be the minimal parabolic subgroup with Lie algebra $\mathfrak{p}$ and put $\mathbf{B} = G/P$ for the maximal flag manifold of G. This flag manifold fibers over the other boundaries of G, which are built from subsets of Σ as follows: Given $\Theta \subset \Sigma$ let $\mathfrak{n}_\mathfrak{f}$ be the subalgebra generated by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \Theta$ and put

$$\mathfrak{p}_\mathfrak{f} = \mathfrak{n}_\mathfrak{f} \oplus \mathfrak{p} .$$

The normalizer P_Θ of $\mathfrak{p}_\mathfrak{f}$ in G is a parabolic subgroup which contains P. The corresponding flag manifold $\mathbf{B}_\Theta = G/P_\Theta$ is the base space for the natural fibration

$\pi_\Theta : \mathbf{B} \to \mathbf{B}_\Theta$ whose fiber is P_Θ/P. We denote by W_Θ the subgroup of the Weyl group generated by the reflections with respect to the simple roots in Θ. A conjugate $Ad(g)H$, $g \in G$, $H \in \mathfrak{a}^+$ is said to be split-regular in $\mathfrak{g}$. Similarly, a split-regular element in G is a exponential $h = \exp(H)$ with $H \in \mathfrak{g}$ split regular. A split-regular $H \in \mathfrak{g}$ belongs to a unique Weyl chamber in $\mathfrak{g}$ (a conjugate of $\mathfrak{a}^+$). If $h_0 \in A^+ = \exp \mathfrak{a}^+$ then h_0 has a finite number of fixed points in $\mathbf{B}$, namely, the base point P and its orbit under M^*. The action of M^* in its orbit factors through M so that the fixed points are given by wP, $w \in W = M^*/M$. The same way, the fixed points in $\mathbf{B}$ of a split-regular $h = gh_0g^{-1}$ with $g \in G$ and $h_0 \in A^+$, are the points gwP. In what follows we say that gwP is the fixed point of type w for h.

We consider now a semigroup $S \subset G$ with $\mathrm{int}(S) \neq \emptyset$. In [11] the control sets for the action of S on the flag manifolds were described by means of the Weyl group W. In this description we have a mapping

$$w \to D_w$$

which associates to $w \in W$ a control set D_w in such a way that the set of transitivity $(D_w)_0$ is the set of the fixed points of type w for the split-regular elements in $\mathrm{int}(S)$. There is just one invariant control set D_1 (see [12] Theorem 3.1) whose set of transitivity is the set of attractors for the split-regular elements in $\mathrm{int}(S)$. The subset defined by

$$W(S) = \Big\{ w \in W \colon\ D_w = D_1 \Big\}$$

is a subgroup of W, and for $w_1, w_2 \in W$, $D_{w_1} = D_{w_2}$ if and only if $w_1 w_2^{-1} \in W(S)$ (see [11] Proposition 4.2). As a consequence one has the main result on the number of control sets in flag manifolds.

Proposition 5. *The number of control sets in a boundary $\mathbf{B}_\Theta = G/P_\Theta$ is the order of the set of double cosets $W(S)\backslash W/W_\Theta$. In particular, the control sets on the maximal boundary $\mathbf{B}$ are in one-to-one correspondence with the cosets in $W(S)\backslash W$.*

Proof: See Corollary 5.2 in [11]. ∎

4 – The number of control sets

Let G be a Lie Group and L a closed subgroup of G. We assume that $Q = G/L$ is a compact homogeneous space of G. We also suppose that S is a subsemigroup of G with $\text{int}_G(S) \neq \emptyset$. Suppose that S acts on Q as a semigroup of diffeomorphisms. In this section we will be interested in determining the number of control sets of S on a compact homogeneous space G/L where L is a closed subgroup of the Lie group G. We start discussing the number of invariant control sets for the action of S on the homogeneous space $Q = G/L$. Since Q is compact there exist a finite number of invariant control sets for S on Q (see [5] Proposition 3.3.8). We denote the number of invariant control sets for S on Q by $\text{ic}(Q)$. We also denote by G_0 the identity component of the Lie group G. We note that since G_0 is an open normal subgroup of G one has that G_0L is open and therefore it is a closed subgroup of G. It is also immediate that $G_0 \cap L$ is a closed subgroup of G. The homogeneous space $G_0/(G_0 \cap L)$ is the connected component of G/L and therefore it is compact. Also, the homogeneous space G/G_0L is the finite set of the components of G/L. We reduce the problem of computing the number of invariant control sets on G/L to the case when G is connected. We denote the number of invariant control sets for S on Q by $\text{ic}(Q)$.

Proposition 6. *Assume that S is a subsemigroup of G with $\text{int}_G(S) \neq \emptyset$ and which generates G in the sense that every element of G is a product of elements of the set $S \cup S^{-1}$. Let G be a Lie group and G/L a compact homogeneous space. Then*

$$\text{ic}(G/L) = \text{ic}(G_0/(G_0 \cap L)\ .$$

Proof: We consider the fiber bundle with projection $G/L \to G/G_0L$ and associated to the principal bundle with projection $G \to G/G_0L$. The typical fiber $G_0L/L = G_0/G_0 \cap L$ of the associated bundle is compact. Since the base space G/G_0L is finite and S generates G we have that S is transitive on G/G_0L. Applying Proposition 2 to this bundle one has that the number of invariant control sets in the total space G/L is equal to the number of invariant control sets in the fiber $G_0/G_0 \cap L$. ■

Although we don't know a similar result for the number of control sets it is reasonable to assume that the Lie group G is connected. Therefore, let G be a connected Lie group with finite dimensional Lie algebra $\mathfrak{g}$. A Levi decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$$

where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semi-simple or $\{0\}$. Let $R = \text{rad}(G)$ (the radical of G) be the connected Lie subgroup of G whose Lie algebra is $\mathfrak{r}$. We know from [10] Proposition 10.12 that R is a closed, solvable and normal Lie subgroup of G. Let H be the semi-simple Lie group generated by $\exp(\mathrm{ad}(\mathfrak{s}))$. It follows from [14] that G decomposes as the semi-direct product

$$G = R \times H \ .$$

It is also shown in [14] Theorem 3.18.13 that if G is simply connected then G is diffeomorphic to the Cartesian product of R with H and decomposes as the direct product

$$G = RH \quad \text{with} \ \ R \cap H = \{1\}$$

where H is closed in G.

From now on, we assume that G is simply connected.

Now, we show that a subsemigroup of a solvable Lie group G with non-empty interior and generating G is transitive on a compact homogeneous space of G.

Lemma 1. *Let G be a solvable Lie group and G/L a compact homogeneous space. Suppose that S is a subsemigroup of G generating it and with $\mathrm{int}_G(S) \neq \emptyset$. Then S is transitive in G/L.*

Proof: We know from [7] Theorem 1.2 that there exists a probability measure μ on G/L which is invariant by the action of G. The result follows from Proposition 6.3 of [13]. ∎

Now, we state the main result of the paper.

Theorem 1. *Let G be a connected and simply connected Lie Group. Suppose that S is a subsemigroup of G generating it and with $\mathrm{int}_G(S) \neq \emptyset$. Take $G = RH$ as a Levi decomposition of G, where R is the radical of G and H is a semi-simple Lie Group. Suppose that RL is closed and that $H \cap RL$ is a parabolic subgroup of H. Then the number of effective control sets of S on G/L is finite and equals to the number of effective control sets of S on the flag manifold $H/(H \cap RL)$. In particular, there is only one invariant control set of S on G/L.*

Proof: Since R is normal in G we have that RL is a closed Lie subgroup of G which contains L. We consider the associated bundle

$$G/L \rightarrow G/RL$$

with typical fiber RL/L. Since G/L is compact we have that G/RL e RL/L are compact homogeneous spaces. Now, we show that the homogeneous space

$G/RL = (RH)/(RL)$ is diffeomorphic to the homogeneous space $H/(H \cap RL)$ of the semi-simple Lie group H. It follows that $(RH)/(RL) = \{h(RL) : h \in H\}$. In fact, an element of $(RH)/(RL)$ is written as $rh(RL)$ with $r \in R$ and $h \in H$. By the normality of R it follows that $rhRL = hRL$. The diffeomorphism between G/RL and $H/(H \cap RL)$ is given by the map

$$\begin{array}{rcl} G/RL & \to & H/(H \cap RL) \\ rhRL & \mapsto & h(H \cap RL) \end{array} .$$

Since the typical fiber $R/(R \cap L)$ is a homogeneous space of a solvable Lie group the Lemma 1 implies that S is transitive on $R/(R \cap L)$. By the Proposition 3 the number of effective control sets on $H/(H \cap RL)$, which is finite, is equal to the number of effective control sets on G/L. Since there is only one invariant control set on a flag manifold of a semi-simple Lie group the Proposition 3 also implies that there is only one invariant control set on G/L. ∎

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