ON THE LOCALLY m-CONVEX ALGEBRA $L_\Gamma(E)$
AND A DIFFERENTIAL-GEOMETRIC INTERPRETATION OF IT

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0 – Introduction

The purpose of the present paper is to supply the algebra $L_\Gamma(E)$ of continuous endomorphisms of a given locally convex space $E$ with a suitable locally $m$-convex topology (Theorem 1.1), a theme already treated by E.A. Michael [4] in a, say, geometric manner by looking at an appropriate local basis of the algebra at issue. We take here, instead, a rather "arithmetic" point of view, by giving a pertinent family $\Gamma = \{p\}$ of submultiplicative seminorms on $L(E)$, starting from a standard family $\Gamma = \{p\}$ of seminorms for the given locally convex space $E$. Apart from the obvious practical utility of the latter point of view, one finds a further justification by getting an improvement of Michael's result, referring to an inverse of the main theorem, as before (cf. Theorem 1.2).

On the other hand, by using this locally $m$-convex algebra structure of $L(E)$, one can define for complete $E$ the exponential function (see e.g. A. Mallios [2]), which can be, of course, at the basis of the differential-geometric interpretation of $L(E) \equiv L_\Gamma(E)$, alluded to at the heading, as well as, of any locally $m$-convex algebra, in general (see Remark 4.1).

1 – The algebra $L_\Gamma(E)$

To start with, suppose that $E$ is a $\mathbb{C}$-vector space and let $p: E \to \mathbb{R}_+$ be a seminorm on $E$. So denoting by $L(E)$ the $\mathbb{C}$-vector space of $\mathbb{C}$-linear endomor-
physms of $E$, one defines the $p$-bound of an element $T \in L(E)$ by the relation

\[
(1.1) \quad p(T) := \sup \left\{ \frac{p(T(x))}{p(x)} : p(x) \neq 0 \right\}
\]

(whenever this exists). We also call $T \in L(E)$, for which $p(T) < +\infty$, $p$-bounded. Thus, we first have the following lemma, whose proof is standard.

**Lemma 1.1.** Keeping the above notation, suppose that $S, T \in L(E)$, with $p(T) < +\infty$. Then, one has the following relations:

1. $p(Tx) \leq p(T) \cdot p(x)$, \quad $x \in E$ ,
2. $p(\lambda T) = |\lambda| \cdot p(T)$, \quad $\lambda \in \mathbb{C}$, \quad $p \in \Gamma$ ,
3. $p(T + S) \leq p(T) + p(S)$ ,
4. $p(T \circ S) \leq p(T) \cdot p(S)$ .

That is, $p$ yields a submultiplicative seminorm on that part of $L(E)$, for which $p < +\infty$, satisfying then (1) as well.

**Proof:** The only thing we have to vindicate is prop. (1) in the case that, for some $x \in E$, $p(x) = 0$. Then, one proves that $p(Tx) = 0$, as well; indeed, if $p(y) \neq 0$, for some $y \in E$, one has (by considering $z_n := x + \frac{1}{n} y$, $n \in \mathbb{N}$):

\[
(1.2) \quad p(Tx) \leq 2 \frac{1}{n} p(T) p(y) , \quad n \in \mathbb{N} ,
\]

which proves the assertion. ■

Of course, prop. (1) in the previous lemma characterizes the $p$-continuity of a $p$-bounded operator $T \in L(E)$. Thus, denoting by

\[
L(E, p) \equiv L_p(E)
\]

the $p$-bounded elements of $L(E)$, one has, by Lemma 1.1, that

\[
L_p(E) \text{ is a locally } m\text{-convex algebra,}
\]

in fact, a seminormed one.

Hence, one obtains.

**Theorem 1.1.** Let $(E, \Gamma \equiv \{p\})$ be a given locally convex space. Then,

\[
(1.5) \quad L(E) \equiv L_E := \bigcap_{p \in \Gamma} L_p(E) \subseteq L(E)
\]
is a unital locally $m$-convex algebra in the topology defined by the family of submultiplicative seminorms (cf. (1.1))

\[(1.6) \quad \Gamma = \{p\}_{p \in \Gamma} \, .\]

In particular, $L_\Gamma(E)$ is (Hausdorff) complete, if $E$ is a complete (Hausdorff) locally convex space.

**Proof:** The first part of the assertion is obviously true, according to (1.4). On the other hand, if $E$ is Hausdorff and $p(T) = 0$, for some $T \in L(E)$, for every $p \in \Gamma$, then by (1.1), $p(Tx) = 0$, for any $p \in \Gamma$ and $x \in E$, so that by hypothesis $Tx = 0$, hence, $L_\Gamma(E)$ is Hausdorff too. Now, suppose that

\[(1.7) \quad (T_\delta)_{\delta \in I} \subseteq L_\Gamma(E) = L(E)\]

is a Cauchy net; viz. for every $\varepsilon > 0$ and $p \in \Gamma$ (cf. (1.6)), there exists $\delta_0(\varepsilon, p) \equiv \delta_0 \in I$, such that

\[(1.8) \quad p(T_\delta - T_{\delta'}) \leq \varepsilon, \quad \text{with} \; \delta, \delta' \geq \delta_0 \, .\]

Thus, for every $x \in E$, one has

\[(1.9) \quad p(T_\delta x - T_{\delta'} x) = p((T_\delta - T_{\delta'}) x) \leq p(T_\delta - T_{\delta'}) \cdot p(x) \leq \varepsilon \cdot p(x)\]

(cf. also (1.1) and Lemma 1.1), that is, $(T_\delta x) \subseteq E$ is a Cauchy net, so that, by the hypothesis for $E$,

\[(1.10) \quad T(x) \equiv \lim_\delta T_\delta x \in E \, .\]

We prove that the operator $T$, as given by (1.10), is $p$-bounded, for every $p \in \Gamma$, viz. $T \in L_\Gamma(E)$ such that $T = \lim_\delta T_\delta$.

Indeed, given $p \in \Gamma$, one has

\[(1.11) \quad p(T_{\delta_0} x - T x) = p(T_{\delta_0} x - \lim_\delta T_\delta x) = \lim_\delta (p(T_{\delta_0} x - T_\delta x)) \equiv \lim_\delta (p(T_{\delta_0} - T_\delta) x) \leq \lim_\delta (p(T_{\delta_0} - T_\delta) \cdot p(x)) = p(x) \cdot \lim_\delta p(T_{\delta_0} - T_\delta) \leq \varepsilon \cdot p(x) \, ,\]

for any $\delta \geq \delta_0(\varepsilon, p)$, as in (1.8). Thus, due to the arbitrariness of $x \in E$ in (1.11), by (1.1),

\[(1.12) \quad p(T_{\delta_0} - T) \leq \varepsilon \, ,\]
hence, $T_{\delta_0} - T \in L_{\Gamma}(E)$, while the same relation (1.12) proves that $T = \lim_{\delta} T_\delta$, as desired, and this terminates the proof. (The relation (1.12) remains true if, in place of $\delta_0$, we put any $\delta' \geq \delta_0$).

Thus, suppose now that $E$ is a (complete) locally convex space so that, according to Theorem 1.1, $L_{\Gamma}(E) \equiv L(E)$ is a (complete) locally $m$-convex algebra. So it admits the so-called Arens–Michael decomposition (cf., for instance, A. Mallios [1, p. 88, Theorem 3.1]), that is one has

\begin{equation}
L(E) = \lim_{\top} L(E)_p ,
\end{equation}

such that one has

\begin{equation}
L(E)_p := \frac{L(E)}{\ker p} ,
\end{equation}

for every $p \in \Gamma$ (see (1.1), (1.6)). On the other hand, the complete locally convex space $E$ is of the form

\begin{equation}
E = \lim_{\top} E_p ,
\end{equation}

where

\begin{equation}
E_p := \frac{E}{\ker p} ,
\end{equation}

for every $p \in \Gamma$. Thus, the latter space being, by definition, a Banach space, one can further consider the Banach algebra

\begin{equation}
L(E_p) , \quad p \in \Gamma .
\end{equation}

In this context, we further remark that one has

\begin{equation}
L(E)_p \subset L(E_p) \subset \phi_p , \quad p \in \Gamma ,
\end{equation}

within a Banach algebra isomorphism (into). Indeed, the said isomorphism (being also a topological one for the respective uncompleted spaces (cf. (1.14) and (1.16)) is given by the relation

\begin{equation}
\tilde{u}_p(x + \ker p) := u(x) + \ker p ,
\end{equation}

for any $u \in L(E)$ and $p \in \Gamma$ (see also Lemma 1.1).

In sum, one has the relation (cf. also (1.13) and (1.18))

\begin{equation}
L(E) = \lim_{\top} L(E)_p = \lim_{\phi_p} \phi_p(L(E)_p) \subset \prod_{p \in \Gamma} L(E_p) .
\end{equation}
For convenience and for later applications, we still note that if

\[(1.21)\]

\[u \in L(E) ,\]

then, by virtue of (1.20), one has

\[(1.22)\]

\[u = \lim_{p \rightarrow \infty} u_p ,\]

where we set (cf. (1.14))

\[(1.23)\]

\[u_p = \phi_p(u + \ker p) , \quad p \in \Gamma .\]

**Scholium 1.1.** Concerning the above topologization of the algebra \(L_\Gamma(E)\), which is already contained in the author’s thesis [6], it was actually conceived independently of \(E\). Michael’s treatment [4] of the same notion, the latter author having given a “geometric” aspect of the same topology, in terms, namely, of neighborhoods of zero. The analytic terminology adopted herewith, as in (1.1), had among other things the advantage of giving a genuine inverse of Theorem 1.1 improving, thus, the analogous result of Michael (ibid. p. 19, Proposition 4.4). See also *Theorem 1.2* below.

As already said, we have the following inverse of Theorem 1.1. Namely, one gets at the next.

**Theorem 1.2.** *Let \((A, \Gamma = (p))\) be a locally \(m\)-convex algebra. Then, there exists a topological algebraic isomorphism of \(A\) into an algebra of the form \(L_\Gamma(E)\), for a locally convex space \(E\).*

**Proof:** By considering the unitization of \(A\), \(A^+ \equiv A \oplus \mathbb{C}\), we first prove our assertion for \(A^+\). Thus, taking now \(E = A^+\), we further look at the (canonical) left representation of \(A^+\), \(x \mapsto \ell(x) \equiv \ell_x, \quad x \in A^+\), such that (cf. (1.1))

\[(1.24)\]

\[\ell_x(p) = p(x) , \quad x \in A^+ .\]

Hence, \(\ell_x \in L(A^+) = L(E)\) (see also (1.5)). On the other hand, one has, of course, the relation

\[(1.25)\]

\[A \subseteq A^+ \subseteq L(A^+) ,\]

within topological algebra isomorphisms, and this completes the proof. \(\blacksquare\)
2 – The exponential function

We have already established in the previous section that \( L(E) \) is a locally \( m \)-convex algebra, for any given locally convex space \( E \), in the topology defined by (1.1), which, moreover, is complete, whenever \( E \) is so.

In this context, it is enough to suppose, for the applications we have in mind, that \( E \) is just \( \sigma \)-complete. Thus, applying an analogous argument as before (cf. Theorem 1.1), one proves that \( L(E) \) is a \( \sigma \)-complete locally \( m \)-convex algebra. Hence, one defines in \( L(E) \) the exponential function, by the relation

\[
\exp u := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot u^n ,
\]

for every \( u \in L(E) \). (In this regard, see also A.Mallios [3: p.196; (8.5)].)

On the other hand, by considering the locally convex space \( (E; \Gamma \equiv \{p\}) \), as before, each one of the respective Banach algebras \( L(E_p) \), \( p \in \Gamma \) (see (1.17)) has a corresponding exponential function \( \exp_p \), given by

\[
\exp_p(u_p) := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot u_p^n ,
\]

which also, for simplicity and by a slight abuse of notation, we denote occasionally, just by \( \exp(u_p) \), for \( u_p \in L(E_p) \). Thus, our next objective is to show that one has (supposing always that \( E \) is \( \sigma \)-complete)

\[
\exp = \lim_{p \to \Gamma} \exp_p .
\]

Indeed, by the very definitions, one proves first that \( (\exp_p)_{p \in \Gamma} \), as given by (2.2), is a projective system of maps, which thus guarantees the existence of the second member of (2.3). Furthermore, the later map is actually \( \exp \), as this is given by (2.1), according to the relation

\[
\exp_p \circ \rho_p = \rho_p \circ \exp_p ,
\]

where \( \rho_p : L(E) \to L(E)_p \), \( p \in \Gamma \), stands for the canonical “projection”, derived from the corresponding Arens–Michael decomposition of \( L(E) \).

3 – Differentiable curves

Suppose we have a locally convex space \( (E; \Gamma \equiv \{p\}) \) and let

\[
\alpha : I \to E
\]

be a curve in \( E \), where \( I \) is an (open) neighborhood of 0 in \( \mathbb{R} \).
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Now, we say that $\alpha$ is \textit{differentiable at} $t_0 \in I$, whenever the following limit exists (in $E$)

\begin{equation}
\dot{\alpha}(t_0) \equiv \lim_{t \to t_0} \frac{1}{t - t_0} \left( \alpha(t) - \alpha(t_0) \right).
\end{equation}

We call (3.2) the \textit{derivative of $\alpha$ at} $t_0$, while we say that $\alpha$ is \textit{differentiable in} $I$, if this happens for every $t \in I$, as before, in which case we denote by $\dot{\alpha} : I \to E$ the corresponding \textit{derivative of $\alpha$}. Of course, as follows from the very definitions, \textit{every differentiable curve $\alpha$}, as above, \textit{is continuous}.

In this context, the following is already an easy consequence of the very definitions.

**Lemma 3.1.** Given a locally convex space $(E, \Gamma \equiv (p))$, a curve $\alpha : I \to E$ is differentiable if, and only if, this is the case for every curve (cf. (1.16))

\begin{equation}
\alpha_p \equiv \pi_p \circ \alpha : I \to E_p, \quad p \in \Gamma,
\end{equation}

where $\pi_p : E \to E_p$ stands for the canonical projection map. $\blacksquare$

As a consequence of the proof of Lemma 3.1, one sees that (cf. (3.3))

\begin{equation}
\dot{\alpha} = (\dot{\alpha}_p \equiv \pi_p \circ \dot{\alpha})_{p \in \Gamma}.
\end{equation}

Thus, we are now in the position to give our main result of this section, that runs as follows.

**Theorem 3.1.** Let $(E, \Gamma \equiv (p))$ be a $\sigma$-complete locally convex space and $L_{\Gamma}(E)$ the corresponding $\sigma$-complete locally \textit{m}-convex algebra, as given by (1.5). Then, for every $u \in L_{\Gamma}(E)$, one finds a solution (curve $\alpha$, cf. (3.1)) of the differential equation

\begin{equation}
\dot{\alpha} = u \alpha
\end{equation}

(where one gets $\dot{\alpha}(t) = u \alpha(t) \equiv u(\alpha(t))$, $t \in I$). This solution is unique, under the requirement that $\alpha(0) = x \in E$.

**Proof:** Based on the Arens–Michael decomposition of the locally \textit{m}-convex algebra $L_{\Gamma}(E)$ (cf. (1.20)), the problem is reduced to the analogous one for the corresponding (for each $p \in \Gamma$) factor Banach algebra $L(E)_p \subseteq_{\phi_p} L(E_p)$.
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(cf. (1.18)): now the solution is obtained in a standard way, through the exponential function (see, for instance, J. Nieto [5: p.29, (1)])). Then, one obtains (see also (3.1))

\[ \alpha(t) = \exp(tu)(x), \quad t \in I, \]

where, of course, \( \alpha(0) = x \in E \). (In this regard, one applies the continuity of the (canonical) projection maps in (1.15), along with (3.2) and the fact that \( \hat{\alpha}(t) \in E \subseteq \lim E_p \) (see (3.4)). The uniqueness is now easily reduced (coordinate-wise) to the factors of the respective Arens–Michael decomposition, where one can then follow J. Nieto [5].)

**Scholium 3.1.** Following an analogous argument, as before, one can further extend, within the present context, several other relevant results of J. Nieto [5]. Thus, we can also discuss solutions of the equation

\[ \hat{\alpha}(t) = u(t) \cdot \alpha(t), \quad t \in I, \]

(cf. also (3.6)). Such type of results are given in [6].

4 – **A differential-geometric interpretation of \( L(E) \)**

Suppose we have again a \( \sigma \)-complete locally convex space \( E \), so that \( L(E) \) (see (1.5)) is a \( \sigma \)-complete locally \( m \)-convex algebra. As the heading indicates, our objective in the sequel is to express the algebra \( L(E) \) as the orbit space of suitable group actions, according to standard differential-geometric arguments.

So we denote by

\[ GL(E) \equiv L(E)^* \subseteq L(E) \]

the group of invertible elements of \( L(E) \), viz. the set of topological vector space automorphisms of the locally convex space \( E \), the same set being, in view of the sort of the topology of \( L(E) \), a topological group. Now, one can consider three types of action of \( GL(E) \); that is, first one has

\[ GL(E) \times GL(E) \to GL(E), \]

such that

\[ (u, v) \mapsto v \circ u \equiv v u = r_u(v), \]

where \( r_u(v) \) denotes the right multiplication of \( v \) by \( u \) in \( GL(E) \).
for any $u, v$ in $GL(E)$, while (4.3) satisfies, of course, the usual axioms for a group action. We thus obtain a topological group action of $GL(E)$ on itself (by right “translations”; similarly, one defines a “left action”). Furthermore, one defines the following topological (viz. continuous) actions of $GL(E)$

\begin{equation}
GL(E) \times L(E) \rightarrow L(E),
\end{equation}

such that

\begin{equation}
(\alpha, u) \mapsto \alpha \circ u \equiv \alpha \cdot u,
\end{equation}

for any $\alpha \in GL(E)$ and $u \in L(E)$, and finally

\begin{equation}
GL(E) \times E \rightarrow E,
\end{equation}

where we define

\begin{equation}
(\alpha, x) \mapsto \alpha(x),
\end{equation}

for any $\alpha \in GL(E)$ and $x \in E$. (It is clear that (4.4) is continuous, being a particular case of (4.2), while the continuity of (4.7) follows easily from Lemma 1.1, prop.(1)).

Now given an element $u \in L(E)$, one obtains the curve in $GL(E) \subseteq L(E)$,

\begin{equation}
\alpha_u : \mathbb{R} \rightarrow GL(E) : t \mapsto \alpha_u(t) := \exp(tu),
\end{equation}

such that, by considering it in $L(E)$, its derivative is given by

\begin{equation}
\dot{\alpha}_u(t) = u \cdot \exp(tu) = u \cdot \alpha_u(t),
\end{equation}

with $t \in \mathbb{R}$, or even (formally)

\begin{equation}
\dot{\alpha}_u = u \cdot \alpha_u.
\end{equation}

**Remark 4.1.** The above argument concerning the algebra $L(E)$, (cf. Theorem 3.1), can be formulated, more generally, for any $\sigma$-complete locally $m$-convex algebra $E$ and its group of invertible elements $E^*$. Thus, based on the preceding Remark 4.1 and the above, we can give the following differential-geometric interpretation of the last argument. That is, one has the next.
Proposition 4.1. Suppose we have a unital $\sigma$-complete locally $m$-convex algebra $E$, with $E^*$ its group of units. Moreover, let

\[(4.11) \quad \text{Hom}^\infty(\mathbb{R}, E^*)\]

be the set of differentiable curves in $E^*$, that are also morphisms of the groups concerned. Then one has

\[(4.12) \quad E = \text{Hom}^\infty(\mathbb{R}, E^*) \equiv T(E^*, 1),\]

within a bijection (the last item in (4.12) denoting the “tangent space” of $E^*$ at $1 \in E^*$).

**Proof:** Given $x \in E$, consider the differentiable curve

\[(4.13) \quad \alpha_x: \mathbb{R} \to E^* : t \mapsto \alpha_x(t) := \exp(tx)\]

(see also, for instance, (4.9)), which is also a morphism of the groups concerned. Furthermore, since $\dot{\alpha}_x(0) = x$ (the same rel. (4.9)), one easily proves that the correspondence $x \mapsto \alpha_x$, as given by (4.13), is one-to-one. On the other hand, based on the uniqueness of the solution curves of (3.5) (see also Remark 4.1, or even (1.25)), one finally proves that the same correspondence is onto, as well, and the proof is complete.

Concerning the above result, we further remark that the same set, as in (4.11), can also be viewed as that one of “left invariant vector fields” on $E^*$. Thus to every $x \in E$, one associates, through the (canonical) right regular representation of $E$, the operator $r_x \in L^1(E)$, which can be thought of as left invariant, according to the relation,

\[(4.14) \quad r_x \circ \ell_y = \ell_y \circ r_x,\]

for any $x, y \in E$. (In this context, we finally note that an analogous study to the preceding one can also be given for non-unital algebras, by employing an appropriate interpretation of the exponential function via the circle operation; see thus, for instance, [7], [8]).

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