ORTHOGONAL POLYNOMIALS AND REGULARLY VARYING SEQUENCES

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ABSTRACT. We introduce a method of estimating asymptotic behaviour of polynomials \( Q_n^{(\alpha)}(x) := \sum_{k \leq n} c_k x^k \), \( n \to \infty \), related to a given polynomial \( Q_n(x) := \sum_{k \leq n} a_k x^k \), where \( (c_k), \) \( k \in N \) is any regularly varying sequence of index \( \alpha \) in the sense of Karamata. Then we apply our results to classical orthogonal polynomials as relevant examples.

Preliminaries

Slowly varying functions \( L(x) \) (s.v.f.) in Karamata’s sense are defined on the positive part of real axis, positive, measurable and satisfying: \( \lim_{x \to \infty} L(\lambda x)/L(x) = 1 \) for each \( \lambda > 0 \). Examples of s.v.f. are:

\[ \ln^a x, \ln^b(\ln x), \exp(\ln^c x), \exp\left(\frac{\ln x}{\ln \ln x}\right), \quad a, b \in R, \quad 0 < c < 1, \quad \text{etc.} \]

A regularly varying function \( R_\alpha(x) \) (r.v.f.) of index \( \alpha \) is defined as \( R_\alpha(x) := x^\alpha L(x), \) \( \alpha \in R. \) An excellent survey of properties, characterization, representation, etc. connected with regular variation is given in [1] and [2]; therefore, we suppose the reader is familiar with it.

In [3] we defined a class \( L^* \) of analytic slowly varying functions, with which we deal afterwards. Namely, for any slowly varying \( L(x) \in \text{Loc}(L) \) (i.e., set of locally bounded functions with a property \( L(0^+) = O(1) \)), we define another s.v.f. \( L^*(x) \in L^* \) by:

\[ L^*(x) := x \int_0^\infty e^{-t} L(1/t) dt, \]

satisfying \( L^*(x) \in C^\infty; L^*(x) \sim L(x), x \to \infty. \) Another remarkable property is the possibility of analytic continuation of \( L^*(x) \) on the right complex half-plane without loss of regularity mode, i.e.:

\[ L^*(z) \sim L^*(|z|) \sim L(|z|), \quad |z| \to \infty, \quad \text{Re} \ z > 0. \]

We need here two more propositions from [3];

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PROPOSITION L4. If \( a(s) \rightarrow \infty, s \rightarrow \infty; a(s) \sim b(s), s \rightarrow \infty, \) then \( R^*(a(s)) \sim R^*_n(b(s)), s \rightarrow \infty. \)

PROPOSITION L5. (smooth variation) The derivatives of analytic regularly varying functions satisfy:

\[
\frac{s^m(R^*_n(s))^{(m)}}{R^*_n(s)} \rightarrow a(a - 1)(a - 2) \cdots (a - m + 1), \quad s \rightarrow \infty, \quad m \in \mathbb{N}.
\]

Also, we have to introduce an operator, defined for polynomials \( Q_n(x) \) with positive coefficients and \( x > 0 \), namely: \( \hat{Q}_n(x) := xQ'_n(x)/Q_n(x). \)

A sequence \( (c_n), n \in \mathbb{N}, \) of positive numbers is regularly varying if:

\[
\lim_{n \to \infty} c_{\lambda n}/c_n = \psi(\lambda) \in (0, \infty), \quad \text{for each } \lambda > 0.
\]

In 1973 Bojanić and Seneta unified the theory of regularly varying functions and regularly varying sequences proving that:

(i) the above limit function \( \psi(\lambda) \) is of the form \( \lambda^\rho \) for some \( \rho \in \mathbb{R}; \)

(ii) the function \( f(x) := c_{[x]} \) varies regularly with index \( \rho \).

Thus we can treat every regularly varying sequence as the integer values of some regularly varying function (the property \( L \in \text{Loc} L \) is obvious here). In the sequel we consider regularly varying sequences \( (c_k), k \in \mathbb{N} \) of index \( \alpha \), generated by an associated regularly varying function, i.e., \( c_k := k^\alpha L(k), \quad k \in \mathbb{N}, \) and, alternatively \( c_k^* := k^\alpha L^*(k), k \in \mathbb{N}, \alpha \in \mathbb{R}. \)

Results

Now, we are able to formulate a crucial theorem concerning asymptotic behaviour of \( Q^*_n(x) := \sum_{k \leq n} c_k^* a_{nk}x^k \), related to a given polynomial: \( Q_n(x) := \sum_{k \leq n} a_{nk}x^k \), with positive coefficients \( a_{nk}. \)

THEOREM A. For any fixed \( x \in R^+ \), if

(I) \[
\sup_n \hat{Q}_n(x) \leq M < \infty,
\]

where the constant \( M \) does not depend on \( x \), and

(II) \[
\lim_n \hat{Q}_n(x)/\phi(n) = a(x) \neq 0
\]

then for some \( \phi \) monotone increasing to infinity with \( n \), then

(A) \[
Q^*_n(x) \sim a^\beta(x)c^*_{[\phi(n)]}Q_n(x), \quad n \to \infty,
\]

for all regularly varying sequences \( (c^*_k), k \in \mathbb{N}, \) of index \( \beta, \beta \leq -1. \)

Then we show, under some supposition about the distribution of zeros of \( Q_n(x) \), that (A) is valid for arbitrary \( \beta \in \mathbb{R} \) (Proposition B2). Finally, using a form of Toeplitz Limits Preservation Theorem we prove that Theorem A is true for any regularly varying sequence \( (c_k), k \in \mathbb{N}. \)

For the proof of the theorem we need some more lemmas.
LEMMA A1. \( \hat{Q}_n(x) > 0 \) for \( x \in \mathbb{R}^+ \).

Proof. Since \( \hat{Q}_n(x) = \frac{1}{Q_n(x)} \left( \frac{xQ'_n(x)}{Q_n(x)} - \hat{Q}_n(x)^2 \right) \) and

\[
x(xQ'_n(x))' = \sum_{k \leq n} k^2 a_{nk} x^k,
\]

we have \( \hat{Q}_n(x) = \frac{1}{Q_n(x)\hat{Q}_n(x)} \sum_{k \leq n} (k - \hat{Q}_n(x))^2 a_{nk} x^k > 0, \), \( x \in \mathbb{R}^+ \).

LEMMA A2. \( \hat{Q}_n(x) \) is monotone increasing with \( x \).


LEMMA A3. Under the condition (I) of Theorem A, for any \( x, t \in \mathbb{R}^+ \),

\[
\frac{Q_n(xe^{-t})}{Q_n(x)} \leq \exp \left( \frac{e^{-Mt} - 1}{M} \hat{Q}_n(x) \right).
\]

Proof. Condition (I) is equivalent with:

(A3.1) \[
\frac{d(\hat{Q}_n(s))}{Q_n(s)} \leq \frac{M}{s} ds, \quad s > 0.
\]

Integrating (A3.1) over \( s \in [xe^{-u}, x], u \geq 0 \), we get: \( \ln \hat{Q}_n(x) - \ln \hat{Q}_n(xe^{-u}) \leq Mu, \)

\( u \geq 0 \), i.e.,

(A3.2) \[
\hat{Q}_n(xe^{-u}) \geq \hat{Q}_n(x)e^{-Mu}.
\]

Integrating (A3.2) over \( u \in [0, t] \), we come to the conclusion of the lemma.

LEMMA A4. Regularly varying sequences \( (c_k^*) \) of index \(-(a + 1), a \geq 0\), have the following integral representation:

\[
c_k^* := \int_0^\infty e^{-kt} u(a, t) dt, \quad k \in \mathbb{N},
\]

where \( u(a, t) \) is given by \( u(a, t) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^t (t - u)^{a-1} L(1/u) du, & a > 0, \\ L(1/t), & a = 0. \end{cases} \)

Proof. Case \( a = 0 \) is valid by definition (L1). For \( a > 0 \) and by the Convolution Theorem for Laplace transform, we get:

\[
\int_0^\infty e^{-kt} u(a, t) dt = \frac{1}{\Gamma(a)} \int_0^\infty e^{-kt} t^{a-1} dt \int_0^\infty e^{-kt} L(1/t) dt
\]

\[
= \frac{1}{k^a} \frac{L^*(k)}{\Gamma(a)} = \frac{L^*(k)}{k^{a+1}} = c_k^*, \quad k \in \mathbb{N}.
\]
Now we are able to give

Proof of Theorem A. Using the expression for \((c^*_n)\) of index \(\beta \leq -1\) from Lemma A4, we have:

\[
\frac{Q_n^*(x)}{Q_n(x)} = \frac{1}{Q_n(x)} \sum_{k \leq n} c^*_n a_{n-k} x^k = \int_0^\infty u(-\beta - 1, t) \frac{Q_n(x e^{-t})}{Q_n(x)} \, dt = \int_0^{\xi_n} (\cdot) + \int_{\xi_n}^\infty (\cdot)
\]

\[= U + W,
\]

where \(\xi_n = \xi_n(x) := \phi(n)^{-1/2}/a(x)\).

For the estimation of expression \(U\) we shall use the following identity:

\[(A.1) \quad \ln \frac{Q_n(x e^{-t})}{Q_n(x)} + t \hat{Q}_n(x) = \int_0^t w \hat{Q}_n(a) \hat{Q}_n(a) \, dw, \quad a := x e^{-t},
\]

which is easy to check by double partial integration. According to Lemma A1 and condition (I) from Theorem A, we have: \(0 < \hat{Q}_n(a) \leq M < \infty\), where the constant \(M\) does not depend on \(a\) or \(n\). Also, since \(a \leq x\), from Lemma A2 follows \(\hat{Q}_n(a) \leq \hat{Q}_n(x)\); hence

\[0 < \int_0^t w \hat{Q}_n(a) \hat{Q}_n(a) \, dw \leq M \hat{Q}_n(x) \int_0^t w \, dw = \frac{M}{2} \hat{Q}_n(x) t^2.
\]

Therefore, from (A.2) we get:

\[\ln \frac{Q_n(x e^{-t})}{Q_n(x)} = \hat{Q}_n(x)(-t + O(t^2)), \quad x \in R^+, \quad t \geq 0;
\]

where the absolute constant in \(O\) does not depend on \(x\) or \(t\). Now,

\[U = \int_0^{\xi_n} \exp \left( \ln \frac{Q_n(x e^{-t})}{Q_n(x)} \right) u(-\beta - 1, t) \, dt = \int_0^{\xi_n} u(-\beta - 1, t) e^{-t \hat{Q}_n(x)} e^{O(t^2 \hat{Q}_n(x))} \, dt.
\]

Since \(e^B = 1 + O(Be^B), \quad B > 0\) with a constant in \(O\) independent of \(B\) and, for \(t \in (0, \xi_n), \quad O(\hat{Q}_n(x) t^2) = O(1)\); from (A.3) we get:

\[U = \int_0^{\xi_n} u(-\beta - 1, t) e^{-t \hat{Q}_n(x)} \, dt + \int_0^{\xi_n} u(-\beta - 1, t) e^{-t \hat{Q}_n(x)} O(t^2 \hat{Q}_n(x)) \, dt
\]

\[= \int_0^{\xi_n} u(-\beta - 1, t) e^{-t \hat{Q}_n(x)} \, dt - \int_{\xi_n}^\infty u(-\beta - 1, t) e^{-t \hat{Q}_n(x)} \, dt
\]

\[+ O(\hat{Q}_n(x)) \int_0^\infty t^2 u(-\beta - 1, t) e^{-t \hat{Q}_n(x)} \, dt = U_1 + U_2 + U_3.
\]

Now, taking into account Proposition L4, condition (II) of Theorem A and elementary properties of regularly varying sequences (cf. [2, pp. 49-54]), we see that:

\[U_1 \sim a^\beta(x) c^*_{\phi(n)}, \quad n \to \infty;
\]
and, evidently:

\[ |U_2| = \int_{\xi_n}^{\infty} e^{-t} u(-\beta - 1, t) e^{-t(\hat{\mathcal{Q}}_n(x) - 1)} dt \]

\[ = O(e^{-\xi_n(\mathcal{Q}_n(x) - 1)}) \int_{0}^{\infty} e^{-t} u(-\beta - 1, t) dt = O(e^{-1/2\phi^{1/2}[n]}), \]

for \( n \) sufficiently large.

Using Proposition we get

\[ U_3 = O(Q_n(x)) \cdot \frac{d^2}{ds^2}(e^s(s))_{s=\mathcal{Q}_n(x)} = O(Q_n(x)) \cdot O\left(\frac{e^s(s)}{s^2}\right)_{s=\mathcal{Q}_n(x)} \]

\[ = O\left(\frac{e^\phi(n)}{\phi(n)}\right). \]

Therefore, we see that: \( U \sim U_1 \sim a^\beta(x)e^\phi(n), \ n \to \infty. \) Estimating \( W \), we consider the polynomial \( P_n(x) := Q_n(x)/x. \) Since \( \hat{\mathcal{Q}}_n(x) = 1 + \hat{P}_n(x), \) from the condition (I) we obtain \( \hat{P}_n(x) \leq 2M \) for \( n \) sufficiently large; hence, Lemma A3 gives:

\[ \frac{P_n(x e^{-t})}{P_n(x)} \leq \exp\left(\frac{e^{-2Mt} - 1}{2M} \hat{P}_n(x)\right). \]

So

\[ W = \int_{\xi_n}^{\infty} e^{-t} u(-\beta - 1, t) \frac{P_n(x e^{-t})}{P_n(x)} dt \leq \int_{\xi_n}^{\infty} e^{-t} u(-\beta - 1, t) \exp\left(\frac{e^{-2Mt} - 1}{2M} \hat{P}_n(x)\right) dt \]

\[ < \exp\left(\frac{e^{-2M\xi_n} - 1}{2M} \hat{P}_n(x)\right) \int_{0}^{\infty} e^{-t} u(-\beta - 1, t) dt = O(e^{-\frac{1}{2}\phi(n)\frac{1}{2}}), \ n \to \infty. \]

Hence, we see that \( Q_n^*(x)/Q_n(x) = U + W \sim a^\beta(x)e^\phi(n), \ n \to \infty, \) and Theorem A is proved.

Analysing Theorem A, we see that condition (I) is the most ambiguous one. It happens that fulfilment of this condition essentially depends on the distribution of the zeros of polynomial \( Q_n(x) \). In this article we are satisfied with the next two propositions:

**Proposition B1.** If all zeros of \( Q_n(x) \) belong to left complex half-plane (including the imaginary axis) then, for all \( x \in \mathbb{R}^4 \), the condition (I) in Theorem A is satisfied with \( M = 2. \)

*Proof.* If \((-z_{nk}), \ k \leq n, \) are the zeros of \( Q_n(x) \), then

\[ Q_n(x) = a_{nm} \prod_{k \leq n} (x + z_{nk}), \ \text{Re} \ z_{nk} \geq 0, \ z_{n1} = 0; \]
i.e.,

\[ \hat{Q}_n(x) = \sum_{k \leq n} \frac{x}{x + z_{nk}} = \sum_{k \leq n} u_{nk}; \quad x \frac{d}{dx} \hat{Q}_n(x) = \sum_{k \leq n} \frac{xz_{nk}}{(x + z_{nk})^2} = \sum_{k \leq n} v_{nk}. \]

Since \( \text{Im} \hat{Q}_n(x) = 0 \), we obtain: \( \hat{Q}_n(x) = \text{Re}(\sum_{k \leq n} u_{nk}) = \sum_{k \leq n} \text{Re} u_{nk} \) and, analogously, \( x \frac{d}{dx} \hat{Q}_n(x) = \sum_{k \leq n} \text{Re} v_{nk} \). But, since for \( x \in R^+ \),

\[ 0 < \frac{\text{Re} v_{nk}}{\text{Re} u_{nk}} = \frac{\text{Re} z_{nk}(x + \text{Re} z_{nk})^2 + \text{Im}^2 z_{nk}(2x + \text{Re} z_{nk})}{(x + \text{Re} z_{nk})((x + \text{Re} z_{nk})^2 + \text{Im}^2 z_{nk})} < 2; \]

we get

\[ x \frac{d}{dx} \hat{Q}_n(x) = \sum_{k \leq n} \text{Re} u_{nk} \frac{\text{Re} v_{nk}}{\text{Re} u_{nk}} < 2 \sum_{k \leq n} \text{Re} u_{nk} = 2 \hat{Q}_n(x), \]

i.e., Proposition B1 is proved.

**Remark.** Because of the nature of Laplace transform, the proof of theorem A holds if the index \( \beta \) of regularly varying sequences \( (c_k^*) \), \( k \in N \), satisfies the condition \( \beta + 1 \leq 0 \).

But, for the special distribution of the zeros of \( Q_n(x) \) mentioned above, we are able to prove that theorem A is valid for all finite values of \( \beta \), i.e.: **Proposition B2.** If all zeros of polynomial \( Q_n(x) \) belong, as before, to the left complex half-plane, then Theorem A is valid for any value of indexes of sequences \( (c_k^*) \).

**Proof.** We consider the polynomial \( R_n(x) := xQ_n(x) = \sum_{k \leq n} ka_{nk}x^k \). The zeros of \( R_n(x) \) are, according to a well-known theorem of Gauss, not outside of the convex polygon determined by the zeros of \( Q_n(x) \); so they are also in the left complex half-plane and condition (I) is satisfied.

**Lemma C2.** If \( \lim_n \phi(n) = +\infty \), then for \( x \in R^+ \), then the following statements are equivalent

\[ \lim_n \frac{\hat{Q}_n(x)}{\phi(n)} = a(x), \quad \lim_n \frac{\hat{R}_n(x)}{\phi(n)} = a(x), \]

**Proof.** It is easy to check that \( \hat{R}_n(x) - \hat{Q}_n(x) = \hat{Q}_n(x) \) i.e., (Lemma A1 and Proposition B1) \( 0 < \hat{R}_n(x) - \hat{Q}_n(x) < 2 \), i.e.,

\[ 0 < \left( \frac{\hat{R}_n(x)}{\phi(n)} - a(x) \right) - \left( \frac{\hat{Q}_n(x)}{\phi(n)} - a(x) \right) < \frac{2}{\phi(n)} \]

wherefrom lemma follows.
Now we can apply Theorem A to the polynomial $R_n(x)$. Remark about zeros of $R_n(x)$ and Lemma C2 says that conditions (I) and (II) are satisfied, so:

$$\sum_{k \leq n} kc_k a_{nk} x^k \sim c^{*}_{[\phi(n)]} a^{\beta}(x) R_n(x) = c^{*}_{[\phi(n)]} a^{\beta}(x) x Q'_n(x)$$

$$\sim \phi(n) c^{*}_{[\phi(n)]} a^{\beta+1}(x) Q_n(x), \quad n \to \infty, \; \beta \leq -1;$$

i.e.,

$$\sum_{k \leq n} kc_k a_{nk} x^k \sim [\phi(n)] c^{*}_{[\phi(n)]} a^{\beta+1}(x) Q_n(x), \quad n \to \infty.$$  

The last relation shows that Theorem A is valid for regularly varying sequences $(c^{*}_n)$ of index $\beta + 1$. Applying the algorithm mentioned above to the polynomial $S_n(x) := x R'_n(x) = \sum_{k \leq n} k^2 a_{nk} x^k$, etc. we come to the conclusion from Proposition B2.

Together, Propositions B1 and B2 produce

**Theorem B.** Let $S$ denote the set of positive reals satisfying the condition (II) of Theorem A, $x_0 \in S$, and let all zeros of the polynomial $Q_n(x) = \sum_{k \leq n} a_{nk} x^k$, $k \in N$, belong to the left complex half-plane (including the imaginary axis). Then, for every fixed $x \in S$, $x > x_0$:

$$Q^{(\alpha)}_n(x) := \sum_{k \leq n} c_k a_{nk} x^k \sim a^{\alpha} c_{[\phi(n)]} Q_n(x), \quad n \to \infty, \; \alpha \in \mathbb{R};$$

where $(c_k)$ is any regularly varying sequence of index $\alpha$.

**Proof.** Propositions B1 and B2 say that Theorem B is valid for the class of sequences $(c^{*}_n)$, in particular

(B1) \quad $Q^{*}_n(x) = \sum_{k \leq n} c^{*}_{nk} a_{nk} x^k \sim a^{\alpha} c^{*}_{[\phi(n)]} Q_n(x), \quad n \to \infty, \; \alpha \in R.$

But $c^{*}_m \sim c_m$, $m \to \infty$ (L2 and L4) so, all we have to prove is $Q^{*}_n(x) \sim Q^{*}_n(x), \quad n \to \infty$, $x_0 < x \in S$. For this purpose we invoke an old proposition:

**Lemma B1.** (O. Toeplitz (1911)) Let the triangular matrix $(p_{nk})$, $k \leq n$, $n \in N$, consist of non-negative elements satisfying $\sum_{k \leq n} p_{nk} = 1$, and let $(s_n)$, $n \in N$ is an arbitrary real sequence. Then necessary and sufficient condition for the implication $\lim_n s_n = s \Rightarrow \lim_n \sum_{k \leq n} s_k p_{nk} = s$ is $\lim_n p_{nk} = 0$, for every fixed $k$.

We are going to use the lemma in the following way. Let

$$p_{nk} := \frac{c^{*}_{nk} a_{nk} x^k}{Q^{*}_n(x)}; \quad s_n := \frac{c_n}{c^{*}_n}, \quad n \in N.$$
Then \( \sum_{k \leq n} p_{nk} = 1; \lim_n s_n = s = 1, \) and
\[
\lim_n \sum_{k \leq n} s_k p_{nk} = \lim_n \frac{\sum_{k \leq n} c_k^* a_{nk} x^k c_k^*}{c_k^*} = \lim_n \frac{Q_n^*(x)}{Q_n^*(x)} = s = 1,
\]
if and only if
\[
(B_2) \quad \lim_n p_{nk} = \lim_n \frac{c_k^* a_{nk} x^k}{Q_n^*(x)} = 0.
\]

To prove this, we recall (Lemma A2) that \( \hat{Q}_n(t) \) is monotone increasing with \( t \), i.e., \( \hat{Q}_n(t) > \hat{Q}_n(x_0) \) for each \( t > x_0 \), i.e.,
\[
\frac{Q_n'(t)}{Q_n(t)} > \frac{\hat{Q}_n(x_0)}{t}, \quad t > x_0.
\]

Integrating the last expression for \( t \in [x_0, x] \), we obtain:
\[
Q_n(x) > Q_n(x_0) (x/x_0)^{\hat{Q}_n(x_0)} > a_n x^k (x/x_0)^{\hat{Q}_n(x_0)}. \quad (B_3)
\]

Since \( x_0, x \in S; \ x/x_0 > 1 \), using \((B_1)\) and \((B_3)\) for fixed \( k \) and sufficiently large \( n \), we get:
\[
p_{nk} = \frac{c_k^* a_{nk} x^k}{Q_n^*(x)} = O\left(\frac{\hat{Q}_n(x_0)}{a_n(x) c_{\lfloor \alpha \rfloor}(n)} (x/x_0)^{\hat{Q}_n(x_0)}\right) = o(1),
\]
and Theorem B is proved.

Now, we shall give some examples concerning classical orthogonal polynomials which are good illustrations for our results.

**Example 1: Laguerre Polynomials.** Laguerre polynomials \( L_n^{(a)}(x) \) of index \( a > -1 \) are given in an explicit form by
\[
L_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-x)^k}{k!},
\]
and all their zeros are real and positive. So, we consider polynomials \( L_n^{(a)}(-x) \), \( x > 0 \), whose zeros are real and negative; hence, they satisfy the conditions of Theorem B. Since
\[
\frac{d}{dx} (L_n^{(a)}(-x)) = \sum_{k=1}^{n} \binom{n+a}{n-k} \frac{x^{k-1}}{(k-1)!} = \sum_{k=1}^{n} \binom{n-1+(a+1)}{n-1-(k-1)} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{n} \binom{n-1+(a+1)}{n-1-k} \frac{x^k}{k!} = L_n^{(a+1)}(-x),
\]
to obtain asymptotic behaviour of $\tilde{L}_n^{(a)}(-x)$, we use Perron’s formula \[4\]

$$L_n^{(a)}(z) = 1/2\pi e^{-1/2} e^{z^2/2} (-z)^{a/2-1/4} n^{a/2-1/4} e^{2\sqrt{-\ln n}} (1 + O(1/\sqrt{n})),$$

for any $z$ in the complex plane cut along the positive part of the real axis. Now, for $z = -x$, $x \in R^+$, after some calculations, we obtain:

$$\tilde{L}_n^{(a)}(-x) = \frac{xL_n^{(a+1)}(-x)}{L_n^{(a)}(-x)} \sim \sqrt{n} x^{\beta/2} \beta [\sqrt{n}]^{\beta} \sqrt{\ln n} L_{n-1}^{(a)}(-x), \quad n \to \infty, \quad x \in R^+,$$

that is: $\lim_n \tilde{L}_n^{(a)}(-x)/\sqrt{n} = \sqrt{x}$, $x > 0$; so, we can apply Theorem B on $Q_n(x) = xL_n^{(a)}(-x)$ with $\phi(n) = \sqrt{n}$, $a(x) = \sqrt{x}$. It follows:

$$\sum_{k=1}^{n} k^\beta L_k \left( \frac{n-1+a}{n-k} \right) \frac{x^k}{(k-1)!} \sim x^{\beta/2} [\sqrt{n}]^{\beta} L_{\sqrt{n}}^{(a)} L_{n-1}^{(a)}(-x), \quad n \to \infty;$$

i.e., putting: $a-1 \to a$, $\beta+1 \to \beta$:

$$\sum_{k=1}^{n} k^\beta L_k \left( \frac{n+a}{n-k} \right) \frac{x^k}{k!} \sim x^{\beta+1/2} [\sqrt{n}]^{\beta-1} L_{\sqrt{n}}^{(a+1)} L_{n-1}^{(a)}(-x), \quad n \to \infty$$

As we already showed $L_n^{(a+1)}(-x) \sim \sqrt{n/x} L_n^{(a)}(-x)$; $n \to \infty$. Hence

**Proposition C.** We have

$$\sum_{k \leq n} c_k \left( \frac{n+a}{n-k} \right) \frac{x^k}{k!} \sim x^{\beta/2} [\sqrt{n}]^{\beta} L_n^{(a)}(-x), \quad x \in R^+, \quad n \to \infty;$$

for any regularly varying sequence $(c_k)$ of index $\beta \in R$.

**Example 2. Jacobi Polynomials.** The Jacobi polynomials $P_n^{(a,b)}(t)$ are given by:

$$P_n^{(a,b)}(t) \sum_{k=0}^{n} \binom{n+a}{n-k} \binom{n+b}{k} \left( \frac{t-1}{2} \right)^k \left( \frac{t+1}{2} \right)^{n-k}, \quad a, b > -1.$$

All their zeros are real and belong to the segment $[-1,1]$. We shall consider the associated class of polynomials $Q_n^{(a,b)}(x)$, defined as:

$$Q_n^{(a,b)} := \sum_{k=0}^{n} \binom{n+a}{n-k} \binom{n+b}{k} x^k = (1-x)^n P_n^{(a,b)} \left( \frac{1+x}{1-x} \right).$$

All their zeros are real and negative, so we can apply Theorem B. It is easy to show that:

$$\frac{d}{dx} Q_n^{(a,b)}(x) = (n+b) Q_n^{(a+1,b)}(x)$$
Therefore,
\[
\lim_{n} \frac{Q^{(a,b)}_n(x)}{n} = \lim_{n} \frac{xQ^{(a+1,b)}_n(x)}{Q^{(a,b)}_n(x)}.
\]

For the estimation of this last expression we use the Darboux formula for the asymptotic behaviour of Jacobi polynomials \([4]\), valid for \(t \notin [-1, 1] \):

\[
P^{(a,b)}_n(t) \sim (t - 1)^{-a/2}(t + 1)^{-b/2}[(t - 1)^{1/2} + (t + 1)^{1/2}]^{a+b} \\
\cdot (2\pi n)^{-1/2}(t^2 - 1)^{-1/4}[t + (t^2 - 1)^{1/2}]^{n+1/2}, \ n \to \infty.
\]

Putting \(t = (1 + x)/(1 - x), \ x > 0\), after some simplifications we get:

\[
Q^{(a,b)}_n(x) = (1 - x)^nP^{(a,b)}_n \left(\frac{1 + x}{1 - x}\right) \sim \frac{(\sqrt{x} + 1)^{a+b+2n+1}}{2\sqrt{\pi n} \ x^{a+1/2}}, \ n \to \infty.
\]

Hence:
\[
\lim_{n} \frac{Q^{(a,b)}_n(x)}{n} = \lim_{n} \frac{xQ^{(a+1,b)}_n(x)}{Q^{(a,b)}_n(x)} = \frac{\sqrt{x}}{1 + \sqrt{x}}, \ x > 0,
\]

i.e., the limit does not depend on the type of Jacobi polynomial \(P^{(a,b)}_n(x)\).

Applying Theorem B to the polynomial \(xQ^{(a+1,b)}_{n-1}(x)\), with: \(a(x) = \sqrt{x}/(1 + \sqrt{x}); \phi(n) = n\), we get (when \(n \to \infty\))

\[
\sum_{k=1}^{n} \frac{(n + a)}{(n - k)} \binom{n - 1 + b}{k - 1} k^a L_k x^k \sim x(1 + x^{-1/2})^{-\alpha} n^a L_n Q^{(a+1,b)}_{n-1}(x), \ x \in R^+,
\]

or, since:

\[
\frac{1}{k} \binom{n - 1 + b}{k - 1} = \frac{1}{n + b} \binom{n + b}{k}, \ k \geq 1; \ x(1 + x^{-1/2})Q^{(a+1,b)}_{n-1}(x) \sim Q^{(a,b)}_n(x),
\]

with \(\alpha\) instead of \(\alpha + 1\),

\[
(D) \quad \sum_{k=1}^{n} \frac{(n + a)}{(n - k)} \binom{n + b}{k} k^a L_k x^k \sim n^a L_n (1 + x^{-1/2})^{-\alpha} Q^{(a,b)}_n(x),
\]

for any slowly varying sequence \((L_k)\) and any \(\alpha \in R\). Since (D) is valid for every \(x > 0\), putting \(x = \frac{(t - 1)}{(t + 1)}\), \(t \notin [-1, 1] \) and multiplying by \( ((t + 1)/2)^n \), we obtain asymptotic behavior in terms of Jacobi polynomials \(P^{(a,b)}_n(t)\):
PROPOSITION E. We have for $n \to \infty$

\[
\sum_{k=1}^{n} c_k \binom{n}{k} \binom{n}{k} \left( \frac{t-1}{2} \right)^k \left( \frac{t+1}{2} \right)^{n-k} \sim c_n \left( 1 + \sqrt{\frac{t+1}{t-1}} \right)^{-\alpha} P_{n,a,b}(t),
\]

for any regularly varying sequence $(c_k)$ of index $\alpha \in \mathbb{R}$, and $t \notin [-1, 1]$.

Analogous formulae for ultraspherical polynomials $P_n^{(\lambda)}(\cdot)$, Legendre’s $(P_n(\cdot))$ and Hermite’s $(H_n(\cdot))$ polynomials can be deduced from (C), (D) and (E) by using identities:

\[
P_n^{(\lambda)}(x) = P_n^{(\lambda-1/2, \lambda-1/2)}(x); \quad P_n(x) = P_n^{(0,0)}(x) \]

\[
H_{2n}(i\sqrt{x}) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(-x); \quad H_{2n+1}(i\sqrt{x}) = (-1)^n 2^{2n+1} n! x L_n^{[1/2]}(-x).
\]

For example:

\[
\sum_{k=1}^{n} c_k \binom{n}{k} \frac{x^k}{x} \sim c_n (1 + x^{-1/2})^{-\alpha} (1 - x)^n P_n \left( \frac{1+x}{1-x} \right), \quad x \in \mathbb{R}^+, \ n \to \infty,
\]

for every regularly varying sequence $(c_k)$ of index $\alpha \in \mathbb{R}$.

References


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