A NOTE ON STABILITY OF MINIMAL SURFACES
IN n-DIMENSIONAL HYPERBOLIC SPACE $H^n(c)$

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Abstract. We improve a result of Barbosa-do Carmo about stability of minimal surfaces in $n$-dimensional hyperbolic space $H^n(c)$.

1. Introduction. In [1] Barbosa and do Carmo obtain the following well-known result:

Theorem 1 [1]. Let $M$ be a minimal surface in an $n$-dimensional hyperbolic space $H^n(c)$. Assume that $D$ is a simply connected compact domain with piecewise smooth boundary on $M$. Let $A$ denote the second fundamental form of $M$. If

$$\int_D (|c| + \frac{|A|^2}{2})\,d\nu < \frac{4\pi}{3},$$

then $D$ is stable.

In this note, we improve the Theorem above as follows:

Theorem 2. Let $M$ be a minimal surface in an $n$-dimensional hyperbolic space $H^n(c)$. Assume that $D$ is a simply connected compact domain with piecewise smooth boundary on $M$. Let $A$ denote the second fundamental form of $M$. If

$$\int_D (\frac{|c|}{5} + \frac{|A|^2}{2})\,d\nu < \frac{4\pi}{3},$$

then $D$ is stable.

Remark. Obviously, our condition (2) is better than condition (1) of Barbosa-do Carmo's.


Key Words and Phrases: minimal surface, stability, hyperbolic space.
2. Preliminaries. Let $H^n(c)$ be an $n$-dimensional simply connected space of constant negative curvature $c$; we also call it the hyperbolic space. Let $M$ be a minimal surface in $H^n(c)$; we denote by $K$ the Gauss curvature of $M$ with respect to the induced metric $ds_M^2$. Let $A$ be the second fundamental form of $M$.

We need the following lemmas to prove Theorem 2.

smallskip Lemma 1. If $M$ be a minimal surface in $H^n(c)$, then

$$|
abla (|A|^2)|^2 \leq 2|A|^2|
abla A|^2.$$  \(3\)

Proof. Let $M$ be a minimal surface $H^n(c)$. By an elementary observation one can see that at each point the dimension of the image of the second fundamental form $A$ of $M$ is at most 2. Thus we may choose $e_3, \ldots, e_n$ so that $h_{ij}^0 = 0$ for all $i, j$ and $\alpha \geq 5$, i.e., we may choose the basis $e_1, e_2, \ldots, e_n$ so that the component $h_{ij}^\alpha$ of $A$ satisfy

$$(h_{ij}^{\lambda}) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h_{ij}^\mu) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h_{ij}^\alpha) = \cdots = (h_{ij}^n) = O,$$  \(4\)

for some functions $\lambda$ and $\mu$. Let $|A|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$ be the square length of the second fundamental form of $M$ and $K_N = \sum_{\alpha,\beta,i,j} R_{\alpha\beta ij}^2$ be the normal scalar curvature of $M$. By (4) and Ricci equation we easily check that $|A|^2 = 2(\lambda^2 + \mu^2)$, $K_N = 16\lambda^2\mu^2$.

Noting $\sum_k (h_{11k}^\alpha)^2 = \sum_k (h_{12k}^\alpha)^2$, $3 \leq \alpha \leq n$, by (4), we have

$$|
abla (|A|^2)|^2 = 4 \sum_k \left( \sum_{i,j,\alpha} h_{ij}^\alpha h_{ij}^\alpha \right)^2$$

$$= 16 \sum_k (\lambda h_{11k}^3 + \mu h_{12k}^4)^2$$

$$\leq 16 \sum_k (\lambda^2 + \mu^2)[(h_{11k}^3)^2 + (h_{12k}^4)^2]$$

$$= 8|A|^2 \sum_k [(h_{11k}^3)^2 + (h_{12k}^4)^2].$$  \(5\)

On the other hand, we have

$$|
abla A|^2 = 2 \sum_{i,k,\alpha} (h_{1ik}^\alpha)^2 = 4 \sum_{k,\alpha} (h_{11k}^\alpha)^2$$

$$\geq 4 \sum_k [(h_{11k}^3)^2 + (h_{12k}^4)^2].$$  \(6\)

We get (3) from (5) and (6). The proof of Lemma 1 is completed.
LEMMA 2. If \( M \) be a minimal surface in \( H^n(c) \), then
\[
\frac{1}{2} \Delta(|A|^2) = |\nabla A|^2 + 2c|A|^2 - \frac{3}{2} |A|^4 + 2(\lambda^2 - \mu^2)^2 \\
\geq |\nabla A|^2 + 2c|A|^2 - \frac{3}{2} |A|^4.
\]

Proof. Denote the matrix \((h^\alpha_{ij})\) by \( H_\alpha \), \( 3 \leq \alpha \leq n \). By Gauss-Codazzi-Ricci equations it was shown in [4] that
\[
\frac{1}{2} \Delta(|A|^2) = \sum_{\alpha, i, j, k} (h^\alpha_{ij})^2 + \sum_{\alpha, i, j, k, l} h^\alpha_{ij} (h^\alpha_{kl} R_{lijk} + h^\alpha_{li} R_{kjk}) \\
+ \sum_{\alpha, \beta, i, j, k} h^\alpha_{ij} h^\beta_{jk} R_{\beta \alpha jk} \\
= |\nabla A|^2 + \sum_{\alpha, \beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha, \beta} (tr(H_\alpha H_\beta))^2 + 2c|A|^2.
\]

By (4), it is easy to check the following formulas
\[
(8) \quad \sum_{\alpha, \beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 = -16\lambda^2 \mu^2, \quad \sum_{\alpha, \beta} (tr(H_\alpha H_\beta))^2 = 4(\lambda^4 + \mu^4).
\]

Substituting (8) into (7), we get
\[
\frac{1}{2} \Delta(|A|^2) = |\nabla A|^2 + 2c|A|^2 - 8(\lambda^2 + \mu^2)^2 + 4(\lambda^4 + \mu^4) \\
= |\nabla A|^2 + 2c|A|^2 - \frac{3}{2} |A|^4 + 2(\lambda^2 - \mu^2)^2.
\]

We completed the proof of Lemma 2.

The following proposition is crucial to prove our Theorem 2.

PROPOSITION 1. Let \( M \) be a minimal surface in \( H^n(c) \), \( ds^2_M \) be the induced metric. Then the Gauss curvature \( \bar{K} \) of the conformal metric \( \sigma ds^2 = \sigma ds^2_M \) satisfies \( \bar{K} \leq 2 \), where
\[
\sigma = \frac{|c|}{5} + \frac{|A|^2}{2} > 0.
\]

Proof. By Gauss equation 2\( K = 2c - |A|^2 \),
\[
(9) \quad \sigma = \frac{|c|}{5} + \frac{|A|^2}{2} = \frac{4c}{5} - K.
\]

Thus we can define a conformal metric \( \bar{\sigma} ds^2 = \sigma ds^2_M \) on \( M \). As it is wellknown, the Gauss curvature \( \bar{K} \) of \( \bar{\sigma} ds^2 \) satisfies (for example, see [2])
\[
(10) \quad -\sigma \bar{K} = -K + \frac{1}{2} \frac{\Delta \sigma}{\sigma} - \frac{|\nabla \sigma|^2}{2\sigma^2},
\]
where $\Delta$ is the Laplacian operator of the metric $ds^2_M$.

By (9) and (10), we get

(11) \[ -\sigma K = \sigma - \frac{4c}{5} + \frac{1}{2} \frac{\Delta \sigma}{\sigma} - \frac{|\nabla \sigma|^2}{2\sigma^2}. \]

By use of Lemma 1, Lemma 2 and (9),

(12) \[ \frac{1}{2} \Delta \sigma = \frac{1}{4} \frac{\Delta (|A|^2)}{\sigma} \geq \frac{1}{2} \frac{|\nabla A|^2 + c|A|^2}{\sigma} - \frac{3}{4} |A|^4 \]

\[ \geq \frac{1}{2} \frac{|\nabla \sigma|^2}{\sigma} - 3\sigma^2 + \frac{4c\sigma}{5}. \]

Combining (11) with (12), we obtain

\[ K \leq 2. \]

We completed the proof of Proposition 1.

3. The Proof of Theorem 2. By use of our Proposition 1, we can prove Theorem 2 in the same way as Barbosa and do Carmo did in [1] for Theorem 1. So we omit the proof of Theorem 2 here.

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References