SEMIPRIME IDEALS OF SKEW POLYNOMIAL RINGS

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Abstract. We study relations among the semiprime ideals of a ring $R$ and those of a skew polynomial ring $S_n$ over $R$, in connection with results obtained in [11] for prime ideals of $R$ and $S_n$.

Notice that results on the prime ideal structure of a skew polynomial ring $R[x, f, d]$ have been obtained only under additional assumptions; Goldie and Mihelč [1] assume that $R$ is right Noetherian and $d = 0$, Jordan [4] assumes that $R$ is right Noetherian and $f$ is the identity map of $R$, while Irving [2 and 3] assumes that $R$ is a commutative ring. In [11] we have studied what happens when $f$ is a non-trivial automorphism of $R$ and $d$ is a non-zero $f$-derivation of $R$ and we have extended these results for skew polynomial rings in finitely many variables over $R$.

1. Preliminaries

All the rings considered in this paper are with identities. Let $R$ be a ring, let $H = \{f_1, \ldots, f_n\}$ be a finite set of automorphisms of $R$ and let $D = \{d_1, \ldots, d_n\}$ be a finite set of mappings from $R$ to $R$, such that $d_i$ is a $f_i$-derivation of $R$, for all $i = 1, \ldots, n$ (i.e. $d_i(a + b) = d_i(a) + d_i(b)$ and $d_i(ab) = ad_i(b) + d_i(a)f_i(b)$, for all $a, b$ in $R$). Then an ideal $I$ of $R$ is called an $H$-ideal if $f_i(I) = I$ for each $i$. Also $I$ is called a $D$-ideal if $d_i(I) \subseteq I$, for each $i$. An ideal $I$ of $R$ which is both an $H$-ideal and a $D$-ideal is called for brevity an $(H, D)$-ideal of $R$. In the special case where $H = \{f\}$ and $D = \{d\}$, $I$ is called an $(f, d)$-ideal of $R$. We recall that an ideal $P$ of $R$ is called a semiprime ideal if, given any ideal $A$ of $R$ such that $A^k \subseteq P$ for some non negative integer $k$, one has $A \subseteq P$. A ring is called semiprime if $0$ is a semiprime ideal. From the previous definition it becomes clear that $R$ is a semiprime ring if and only if $R$ has no non zero nilpotent ideals. Now an $(H, D)$-ideal of $R$ is called an $(H, D)$-semiprime ideal if, given any $(H, D)$-ideal $A$ of $R$ such that $A^k \subseteq I$ for some non negative integer $k$, one has $A \subseteq I$, and $R$ is called an $(H, D)$-semiprime ring if $0$ is an $(H, D)$-semiprime ideal of $R$. The notions

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of an $H$-semiprime and a $D$-semiprime ideal of $R$ can be also defined in the obvious way. Assume next that $d_i \circ d_j = d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $d_i \circ f_j = f_j \circ d_i$ for all $i, j = 1, \ldots, n$ and consider the set $S_n$ all polynomials in $n$ variables $x_1, \ldots, x_n$ over $R$. Define in $S_n$ addition in the usual way and multiplication by the relations $x_i r = r x_i$ and $x_i x_j = x_j x_i$, for all $r \in R$ and all $i, j = 1, \ldots, n$. Then $S_n$ is an Ore extension over $S_{n-1}$ (cf. [7]) for all $i = 1, \ldots, n$, where $S_0 = R$ (cf. Theorem 2.4 of [9]). We call the ring $S_n$ a skew polynomial ring in $n$ variables over $R$ and we denote it by $S_n = R[x_1, f_1, d_1] \ldots [x_n, f_n, d_n]$. Notice that under these conditions one can extend $f_i$ to an automorphism and $d_i$ to an $f_i$-derivation of $S_n$ by putting $f_i(x_j) = x_j$ and $d_i(x_j) = 0$ for all $i, j = 1, \ldots, n$ (cf. Theorems 2.2 and 2.3 of [9]). In the special case where $f_i$ is the identity for all $f_i$ in $H$ we get the skew polynomial ring $S_n^* = R[x_1, d_1] \ldots [x_n, d_n]$, while if $d_i = 0$ for all $d_i$ in $D$ we get the skew polynomial ring $S_n' = R[x_1, f_1] \ldots [x_n, f_n]$. When $R$ is right Noetherian the usual proof of the Hilbert’s Basis Theorem adapts easily to show, together with induction on $n$, that $S_n$ is a right Noetherian ring too (this is not true if we take $H$ to be any set of monomorphisms of $R$, cf. [6]).

2. Main results

2.1. Theorem. Let $P$ be an $H$-semiprime ideal of $S_n$; then $P \cap R$ is a $(H, D)$-semiprime ideal of $R$.

Proof. Since $P$ is a $H$-ideal of $S_n$, $P \cap R$ is an $H$-ideal of $R$. On the other hand, for all $r \in P \cap R$ and each $i$, $d_i(r) = x_i r - f_i(r) x_i$, is in $P \cap R$, therefore $P \cap R$ is an $(H, D)$-ideal of $R$. Now, let $A$ be any $(H, D)$-ideal of $R$, such that $A^k \subseteq P \cap R$ for some non negative integer $k$. Then $AS_n$ is an ideal of $S_n$, because $x_i A \subseteq f_i(A) x_i + d_i(A) \subseteq A x_i + A \subseteq AS_n$ for each $i$. It is also clear that $AS_n$ is an $H$-ideal of $S_n$. But $(AS_n)^2 = A(AS_n S_n) \subseteq A^2 S_n$ and therefore an easy induction shows that $(AS_n)^k \subseteq A^k S_n$. Thus $(AS_n)^k \subseteq (P \cap R) S_n \subseteq P$. Then, by our hypothesis, $AS_n \subseteq P$ and therefore $A \subseteq AS_n \cap R \subseteq P \cap R$, and this finishes the proof.

We record two obvious corollaries.

2.2. Corollary. Let $P$ be a semiprime ideal of $S_n^*$, then $P \cap R$ is a $D$-semiprime ideal of $R$.

2.3. Corollary. Let $P$ be an $H$-semiprime ideal of $S_n^*$, then $P \cap R$ is an $H$-semiprime ideal of $R$.

2.4. Theorem. Let $R$ be a right Noetherian ring and let $P$ be a semiprime ideal of $S_n^*$ none of whose minimal primes contains $x_i$, for each $i$. Then $P \cap R$ is an $H$-semiprime ideal of $R$.

Proof. It is enough to show that that $P$ is an $H$-ideal of $S_n^*$ and then to apply Corollary 2.3.
For this, since $S_n$ is right Noetherian, there exist finitely many prime ideals of $S_n$, say $P_1, \ldots, P_k$, such that $P_1 \ldots P_k \subseteq P$ and $P_1, \ldots, P_k \subseteq P$. Thus $P_1 \cap P_2 \cap \ldots \cap P_k \subseteq P$. But, $(P_1 \cap P_2 \cap \ldots \cap P_k)^k \subseteq P_1 P_2 \ldots P_k \subseteq P$, therefore $P_1 \cap P_2 \cap \ldots \cap P_k \subseteq P$ and so $P_1 \cap P_2 \cap \ldots \cap P_k = P$.

Choose $P'$ to be any of the above mentioned prime ideals of $S_n$ and let $g$ be in $P$. Then $g$ is also in $P'$ and therefore $f_i(g)x_i = x_ig$, is in $P'$, for each $i$. Thus $f_i(g)S_n x_i = f_i(g)x_iS_n \subseteq P'$. But $x_i$ is not in $P'$, therefore $f_i(g)$ is in $P'$ and so $f_i(g) \subseteq P$. Thus $f_i(P) \subseteq P$.

Now we have the ascending chain of ideals $P \subseteq f_i^{-1}(P) \subseteq f_i^{-2}(P) \subseteq \ldots$, which by the Noetherian property becomes stable after a finite number of steps, say $m$. Then $f_i^{-m}(P) = f_i^{-m-1}(P)$, therefore $f_i(P) = P$ and this finishes the proof.

Jordan [5] has shown that if $I$ is a d-prime ideal of $R$, then $IS_1^*$ is a prime ideal of $S_1^*$. The following theorem shows that the same result holds if we replace the term “prime” with the term “semiprime” in the more general situation of the skew polynomial ring $S_1$.

2.5. Theorem. Let $I$ be a $(f_1, d_1)$-semiprime ideal of $R$, then $IS_1$ is an $f_1$-semiprime ideal of $S_1$.

Proof. It is easy to check that $IS_1$ is an $f_1$-ideal of $S_1$ (cf. the proof of Theorem 2.1 for the ideal $A S_n$). Next let $A$ be an $f_1$-ideal of $S_1$ such that $A^k \subseteq IS_1$ for some non-negative integer $k$. Denote by $T(A)$ the set of all leading coefficients of the elements of $A$, which are polynomials in $x_1$ with coefficients in $R$. We are going to show that $T(A)$ is an $(f_1, d_1)$-ideal of $R$. For this let $g$ and $h$ be any elements of $A$ with degrees $m$ and $l$ and leading coefficients $a$ and $b$ respectively. Without loss of the generality we may assume that $m \geq l$. Then $a \pm b$ is either zero or the leading coefficient of $g \pm h x_1^{m-1}$ which is in $A$, therefore $a \pm b$ is in $T(A)$. Moreover, for any $r$ in $R$, $ar$ is either zero or the leading coefficient of $rg$. On the other hand $g f_1^{-m}(r) = a x_1^m f_1^{-m}(r) + \text{terms of lower degree} = ar x_1^m + \text{terms of lower degree}$, therefore $ar$ is either zero or the leading coefficient of $g f_1^{-m}(r)$. Thus $ar$ is in $T(A)$ and therefore $T(A)$ is an ideal of $R$. Furthermore $f_1(a)$ and $f_1^{-1}(a)$ are the leading coefficients of $f_1(g)$ and $f_1^{-1}(g)$ respectively. Finally, if $g = \sum a_m x_1^m$, with $a_m = a$, then

$$x_1 g - f_1(g)x_1 = \sum_{i=0}^m [x_1 a_i - f_1(a_i)x_1] x_1^i = \sum_{i=0}^m d(a_i) x_1^i,$$

therefore $d(a)$ is either zero or the leading coefficient $x_1 g - f_1(g)x_1$ which is in $A$. Thus $T(A)$ is in fact an $(f_1, d_1)$-ideal of $R$. Also, since $a x_1^m f_1^{-m-1}(b) x_1^i = ab x_1^{m+1} + \text{terms of lower degree}$, $ab$ is either zero or the leading coefficient of $a x_1^m f_1^{-m-1}(b) x_1^i$. Thus $ab$ is in $T(A^2)$ and therefore $[T(A)]^2 \subseteq T(A^2)$. Then an easy induction shows that $[T(A)]^k \subseteq T(A^k)$, for any non-negative integer $k$. Thus $[T(A)]^k \subseteq T(IS_1) = I$ and therefore, by our hypothesis,

$$T(A) \subseteq I.$$ (1)
Now we may as well assume that $A \supset IS_1$, otherwise we can use $A + IS_1$ instead of $A$. This, together with the relation (1), shows that if $g = \sum_{i=0}^{m-1} a_i x_1^i$ is in $A$, then $a_m x_1^m$ is in $A$ and therefore $g - a_m x_1^m = \sum_{i=0}^{m-1} a_i x_1^i$ is in $A$. Thus $a_{m-1}$ is in $T(A) \subseteq I$. Repeating this argument we finally get that $a_i$ is in $I$ for each $i = 0, 1, \ldots, m$ and therefore $A \subseteq IS_1$, as required.

We are going now to show that an analogous result holds for the skew polynomial ring $S_n$. For this we need the following lemma.

2.6. Lemma. Let $A$ be an $H$-ideal of $S_n$ and let $T(A)$ be the set of all leading coefficients of the elements of $A$ written as polynomials in $x_n$ with coefficients in $S_{n-1}$. Put $T_i(A) = T(T_{i+1}(A))$ in $S_i$, for each $i = 0, 1, \ldots, n-1$, where $S_0 = R$ and $T_n(A) = A$. Then $T_0(A)$ is an $(H, D)$-ideal of $R$ and $[T_0(A)]^k \subseteq T_0(A^k)$ for any non-negative integer $k$.

Proof. Since $T_{n-1}(A) = T[T_{n-1}(A)] = T(A)$, $T_{n-1}(A)$ is an $(f_n, d_n)$-ideal of $S_{n-1}$. Similarly $T_{n-1}(A) = T[T_{n-1}(A)]$ is an $(f_n, d_{n-1})$-ideal of $S_{n-2}$. Now let $r = r(x_{1}, \ldots, x_{n-1})$ be any element of $T_{n-1}(A)$; then there exist $s$ in $T_{n-1}(A)$ with leading coefficient $r$ with respect to $x_{n-1}$. Then $f_n(s), f_{n-1}(s)$ and $r_{n}(s)$ are all in $T_{n-1}(A)$, while $f_n(x_{n-1}) = x_{n-1}$ and $d_n(x_{n-1}) = 0$; therefore $f_{n}(r), f_{n-1}(r)$ and $d_{n}(r)$ are all in $T_{n-2}(A)$. Thus $T_{n-2}(A)$ is also an $(f_n, d_n)$-ideal of $S_{n-2}$. Repeating this procedure we shall eventually find that $T_0(A)$ is an $(H, D)$-ideal of $R$, as required. In the proof of Theorem 2.5 we have seen that $[T(A)]^k \subseteq T(A^k)$, therefore $[T_{n-1}(A)]^k \subseteq T_{n-1}(A^k)$. Next, applying induction on $m$, we assume that $[T_{n-m}(A)]^k \subseteq T_{n-m}(A^k)$. Then

$$[T_{n-m-1}(A)]^k = [T[T_{n-m}(A)]]^k \subseteq [T[T_{n-m}(A^k)]]^k \subseteq T[T_{n-m}(A^k)] = T_{n-m-1}(A^k).$$

Thus, if we put $m = n$, we get that $T_0(A)^k \subseteq T_0(A^k)$ and this finishes the proof.

We are ready now to prove

2.7. Theorem. Let $I$ be an $(H, D)$-semiprime ideal of $R$; then $IS_n$ is an $H$-semiprime ideal of $S_n$.

Proof. Proceeding as in the proof of Theorem 2.1 for the ideal $A S_n$ one can show that $IS_n$ is an $H$-ideal of $S_n$. Next let $A$ be an $H$-ideal of $S_n$ such that $A^k \subseteq IS_n$. Then, by Lemma 2.6, $T_0(A)$ is an $(H, D)$-ideal of $R$ and $[T_0(A)]^k \subseteq T_0(A^k) \subseteq T_0(ISA_n) = I$; therefore $T_0(A) \subseteq I$. Without loss of generality we may assume that $A \supseteq IS_n$, otherwise we may use $A + IS_n$ instead of $A$. Let $g = \sum_{i=0}^{m_1} a_i x_n^i$ be a polynomial of $A$ with coefficients in $S_{n-1}$; then $a_{m_1}$ is in $T_{n-1}(A)$. We write $a_{m_1} = \sum_{i=0}^{m_2} b_i x_n^i$. Then $b_{m_2}$ is in $T_{n-2}(A)$ and $a_{m_1} x_n^{m_1} + \sum_{i=0}^{m_2} b_i x_n^i x_n^{m_2} = \sum_{i=0}^{m_3} a_i x_n^i$. We proceed the same way until we find some $r$ in $T_0(A)$, such that $r = r x_n x_n^2 \ldots x_n^{m_3}$ is term of $a_{m_3} x_n^{m_3}$. Then $r$ is in $I$, therefore $h$ is in $IS_n \subseteq A$. Thus $g - h$ is in $A$. Repeating the same argument for $g - h$ and keeping going in the same way we eventually find that $a_{m_1} x_n^{m_1}$ is in $IS_n \subseteq A$. Thus $\hat{g} = g - a_{m_1} x_n^{m_1}$ is in $A$. If we apply the same
argument for \( \tilde{g} \) and we keep going in the same way we finally find that \( g \) is in \( IS_n \), which was to be proved.

The following are straightforward corollaries of Theorems 2.1 and 2.7.

2.8. Corollary. The skew polynomial ring \( S_n \) is \( H \)-semiprime if and only if \( R \) is an \((H,D)\)-semiprime ring.

2.9. Corollary. If \( I \) is a \( D \)-semiprime ideal of \( R \), then \( IS_n^* \) is a semiprime ideal of \( S_n^* \), therefore \( S_n^* \) is a semiprime ring if and only if \( R \) is a \( D \)-semiprime ring.

2.10. Corollary. If \( I \) is an \( H \)-semiprime ideal of \( R \), then \( IS_n^I \) is an \( H \)-semiprime ideal of \( S_n^I \), therefore \( S_n^I \) is an \( H \)-semiprime ring if, and only if \( R \) is an \( H \)-semiprime ring.

3. Remarks

(1) An \((H,D)\)-ideal \( I \) of \( R \) is called an \((H,D)\)-prime ideal if, given any two \((H,D)\)-ideals \( A \) and \( B \) of \( R \) such that \( AB \subseteq I \), one has either \( A \subseteq I \) or \( B \subseteq I \). The notions of an \( H \)-prime and of a \( D \)-prime ideal of \( R \) can be also defined in the obvious way.

The statements of all results obtained in the previous section remain true if we replace the term “semiprime”, whenever it appears, with the term “prime” (cf. [11]).

(2) The statement of Theorem 2.4 should be restated as follows: “Let \( P \) be a prime ideal of \( S_n^I \) such that \( x_i \) is not in \( P \), for each \( i \). Then \( P \cap R \) is an \( H \)-prime ideal of \( R \)”.

That is, the ring \( R \) need not be right Noetherian in this case. But the hypothesis that \( x_i \) is not in \( P \) for each \( i \) is not superfluous (cf. [1, Example 3]).

(3) If \( R \) is a right Noetherian ring and \( I \) is an \( H \)-prime ideal of \( R \), then \( IS_n^I \) is a prime ideal of \( S_n^I \) (cf. [11, Theorem 2.9]). Attempts to prove that the above mentioned result remains true for the ring \( R \) which is not right Noetherian, i.e. that we can replace the term “prime” with the term “semiprime” and thus to produce a result stronger than Corollary 2.10, have proved unsuccessful.

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