A CHARACTERIZATION OF FORMALLY SYMMETRIC UNBOUNDED OPERATORS

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Abstract. We give necessary and sufficient conditions for an operator in a Hilbert space to be formally symmetric, symmetric or self-adjoint. This generalizes the well-known fact that a bounded operator $T$ is self-adjoint if and only if $T^*T \leq (\text{Re } T)^2$. The proof is based on a well-behaved extension of the corresponding symmetric operator.

0. Introduction

Fong and Istratescu [1] and also Kittaneh [2] have proved the following:

Theorem A. A bounded operator $T$ is self-adjoint if and only if $T^*T \leq (\text{Re } T)^2$.

They used Theorem A to investigate some classes of bounded operators — $\theta, WN$ and hyponormal operators. A large number of well-known and important operators, for example $x + i \frac{d}{dx}$, belongs to similar classes of unbounded operators. The aim of this note is to extend Theorem A to unbounded operators and to make it suitable for dealing with such situations. Our main result is Theorem 1 in which we present characterizations for an operator to be formally symmetric, symmetric or self-adjoint (Theorems 2, 3).

1. Preliminaries

Suppose that $(H, \langle \cdot | \cdot \rangle)$ is a separable, complex, infinite dimensional Hilbert space and let $(H \oplus H, \langle \cdot | \cdot \rangle)$ denote the usual product space. Throughout this paper we assume that all operators are linear. Let $D(A)$ denote the domain of an operator $A$. The operators $(A+A^*)/2$ and $(A+A^*)/2i$ (with $\Delta(A) = D(A) \cap D(A^*)$ as their domains) will be denoted by $\text{Re } A$ and $\text{Im } A$ respectively. If $A$ is a restriction of $B$ on $D(A)$, we will write $A \subset B$. Whenever $\Delta(A)$ is dense in $H$, we will denote the domains of $(\text{Re } A)^*$ and $(\text{Im } A)^*$ by $D(\text{Re } A)^*$ and $D(\text{Im } A)^*$ respectively. We recall

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that a densely defined operator \( A \) is said to be symmetric iff \( \langle Ax|y \rangle = \langle x|Ay \rangle \) for all \( x, y \in D(A) \), i.e. if \( A \subset A^* \). It is said to be formally symmetric iff \( A^*x = Ax \) for all \( x \in \Delta(A) \) i.e. iff \( \text{Im } A \subset 0 \). Note that \( \text{Re } A \) and \( \text{Im } A \) are symmetric whenever \( \Delta(A) \) is dense in \( H \).

2. The construction

**Lemma 1.** For a closed, symmetric operator \( A \) in \( H \) we define the operator \( A^* \) by \( A^*(x,y) = (Ax,Ay) \). If the domain of \( A^* \) is given by \( D(A^*) = \{(x,y) \in D(A) \times D(A^*); x - y \in D(A)\} \) then \( A^* \) is one self-adjoint extension of \( A \oplus (-A) \).

**Proof.** For all \((x,y)\) and \((f,g)\) in \( D(A) \) we have that

\[
\langle A^*(x,y)|f,g \rangle = \langle A^*x|f \rangle - \langle A^*y|g \rangle = \langle A^*(x - y)|f \rangle + \langle A^*y|(f - g) \rangle.
\]

Since \( x - y \) and \( f - g \) are in \( D(A) \), it follows that

\[
\langle A^*(x - y)|f \rangle + \langle A^*y|(f - g) \rangle = \langle A(x - y)|f \rangle + \langle y|A(f - g) \rangle
\]

\[
= \langle (x,y)|A^*(f,g) \rangle.
\]

So \( A^* \) is symmetric.

Suppose that \( \lim_{n \to \infty} (x_n,y_n) = (x,y) \) and \( \lim_{n \to \infty} (A^*x_n,-A^*y_n) = (u,v) \) for some \( (x_n,y_n) \in D(A) \) and some \( x, y, u, v \in H \). This implies that \( \lim_{n \to \infty} (x_n - y_n) = x - y \) and \( \lim_{n \to \infty} A(x_n - y_n) = \lim_{n \to \infty} A^*(x_n - y_n) = u + v \). Since \( A^* \) and \( A \) are closed and \( x_n - y_n \in D(A) \), it follows that \( x - y \in D(A) \) and \( x, y \in D(A^*) \). Moreover, \( A^*x = u \) and \( A^*y = -v \). Therefore \( (x,y) \in D(A) \) and also \( A^*(x,y) = (A^*x - A^*y) = (u,v) \) is closed.

Finally, suppose that \( (x,y) \in R(A^* + iI) \). Then it follows that \( \langle x|(A^* + iI)f \rangle = \langle y|(A^* - iI)g \rangle \) for all \( (f,g) \in D(A) \) and, in particular \( \langle x|(A^* + iI)f \rangle = 0 \) for all \( f \in D(A) \). Therefore \( x \in (A^{**}) = D(A) \) and, moreover, \( x \in \text{Ker}(A - iI) \).

It now follows that \( 2||x||^2 = \langle (A + iI)x|x \rangle = \langle x|(A - iI)x \rangle = 0 \), and hence \( x = 0 \). Analogously, we can prove that \( y = 0 \) and thus \( R(A^* + iI) = \{0\} \). The equality \( R(A^* - iI) = \{0\} \) follows similarly, and hence \( A^* \) is self-adjoint.

**Remark 1.** An alternative proof of Lemma 1 can be obtained by using von Neumann’s formulae for self-adjoint extensions of \( A \oplus (-A) \). The corresponding partial isometry \( V \) is given by

\[
V(x,y) = -(y,x), \text{ for all } (x,y) \in \text{Cl } (R(A \oplus (-A) + iI)) ,
\]

\[
V(x,y) = 0, \text{ for all } (x,y) \in \text{Ker } (A^* \oplus (-A^*) - iI) .
\]

**Lemma 2.** Let \( A \) and \( B \) be closed symmetric operators and assume that \( D(A) \subset D(B) \) and \( D(A^*) \subset D(B^*) \). Then there exist selfadjoint extensions \( A^* \) and \( B^* \) of \( A \oplus (-A) \) and \( B \oplus (-B) \) respectively, such that \( D(A^*) \subset D(B^*) \).

**Proof.** It is sufficient to take the extension constructed in Lemma 1. Then the required inclusion can be shown by a straightforward computation.
3. Main results

**Theorem 1.** Let $A$ and $B$ be symmetric operators and assume that $D(A) \subset D(B)$, $D(A^*) \subset D(B^*)$ and also
\[ \|(A^* - iB^*)x\| \leq ||A^*x|| \] (a)
for all $x \in D(A^*)$. Then $B \subset 0$.

**Proof.** Without loss of generality we may assume that $A$ and $B$ are closed. To see this, note that (a) implies $\|Bx\| \leq 2\|Ax\|$ for all $x \in D(A)$ and hence $D(A^-) \subset D(B^-)$. Because of $A^* = A$ and $B^* = B^*$ it follows that $D(A^*) \subset D(B^*)$ and $\|(A^* - iB^-)x\| \leq ||A^*x||$ for all $x \in D(A^*)$. So, according to Lemma 2, let $A$ and $B$ be the corresponding self-adjoint extensions of $A \oplus (-A)$ and $B \oplus (-B)$, respectively. A simple calculation gives
\[ \|(A - iB)(x, y)\| \leq ||A(x, y)|| \] (a')
for all $(x, y) \in D(A)$. Let $E$ be the spectral measure induced by $A$ and let $\gamma \in \delta \subset \mathbb{R}$, for some measurable bounded set $\gamma$ and $\delta$. We define $A(\delta) = E(\delta)AE(\delta)$ and $B(\delta) = E(\delta)BE(\delta)$. Since $E(\delta)h \in D(A)$, it follows by Lemma 2 that $E(\delta)h \in D(B)$, for an arbitrary $h \in H \oplus H$. Hence $D(B(\delta)) = H \oplus H$. Obviously $B(\delta)$ is symmetric and therefore self-adjoint. Then there exists a sequence $\{h_n\}_{n \in \mathbb{N}}$ of unit vectors in $H \oplus H$ such that $\lim_{n \to \infty} (B(\delta) - \lambda)h_n = 0$ for some $\lambda \in \mathbb{R}$ satisfying $|\lambda| = \|B(\delta)\|$. It follows from (a') that
\[ \|B(\delta)h_n\| \leq -2 \operatorname{Re} \langle A(\delta)h_n, (B(\delta) - \lambda)h_n \rangle . \] (a'')
Letting $n \to \infty$ we get $\|B(\delta)\|^2 \leq 0$, and consequently $E(\delta)BE(\delta) = 0$. Since $\gamma \subset \delta$ we conclude that $E(\delta)BE(\gamma) = 0$. If $\bigcup \{\gamma_n: n \in \mathbb{N}\} = \bigcup \{\delta_n: n \in \mathbb{N}\}$ for some increasing sequences $\{\gamma_n\}_{n \in \mathbb{N}}$ and $\{\delta_n\}_{n \in \mathbb{N}}$, it follows that $BE(\gamma) = s\lim_{n \to \infty} E(\delta_n)BE(\gamma) = 0$ because $s\lim E(\delta_n) = I$. Moreover, $s\lim_{n \to \infty} E(\gamma_n) = I$ implies $B = s\lim_{n \to \infty} BE(\gamma_n) = 0$, since $B$ is closed. Consequently, $B \subset 0$ as required.

**Remark 2.** If, in addition, $A$ is (essentially) self-adjoint, then the assumption $D(A^*) \subset D(B^*)$ can be omitted and the proof of Theorem 1 simplified. Also, the use of lemmas becomes unnecessary.

As a consequence of Theorem 1, we give the following characterization.

**Theorem 2.** If $\Delta(T)$ is dense in $H$, then $T$ is formally symmetric if and only if:
(1) $D(\operatorname{Re}T)^* \subset D(\operatorname{Im}T)^*$, \hspace{0.5cm} (2) $\|(\operatorname{Re}T)^*x - i(\operatorname{Im}T)^*x\| \leq ||(\operatorname{Re}T)^*x||$ for all $x \in D(\operatorname{Re}T)^*$.

**Proof.** If (1) and (2) are true, then $\operatorname{Im}T \subset 0$ by Theorem 1, and hence $T$ is formally self-adjoint. The necessity of (1) is obvious.

**Lemma 3.** If $D(T) \subset D(T^*)$ for an operator $T$, then the following are equivalent:
(1) $D(\operatorname{Re}T)^* \subset D(\operatorname{Im}T)^*$; \hspace{0.5cm} (1') $D(\operatorname{Re}T)^* \subset D(T^*)$. 
If the assumption (1) is satisfied, then \( T^* x = (\text{Re} T)^* x - i(\text{Im} T)^* x \) for every \( x \in D(\text{Re} T)^* \).

**Proof.** Since \( D(\text{Re} T) = D(\text{Im} T) = D(T) \) it follows that \( D(\text{Re} T)^* \cap D(\text{Im} T)^* \subset D(T^*) \) and \( D(\text{Re} T)^* \cap D(T^*) \subset D(\text{Im} T)^* \) and therefore the equivalence of (1) and (1') is obvious. Because of \( T = \text{Re} T + i \text{Im} T \) it follows that 
\[ T^* \supset (\text{Re} T)^* - i(\text{Im} T)^* \]
from which we derive the rest of the statement.

**Theorem 3.** An operator \( T \) is symmetric (resp. self-adjoint) iff
\[
\begin{align*}
(0') & \quad D(T) \subset D(T^*), \quad (\text{resp. } D(T) = D(T^*)) \\
(1') & \quad D(\text{Re} T)^* \subset D(T^*); \\
(2') & \quad ||T^* x|| \leq ||(\text{Re} T)^* x||
\end{align*}
\]
for all \( x \in D(\text{Re} T)^* \).

**Proof.** If (0'), (1') and (2') are true, then \( T \) is formally symmetric by Lemma 3 and Theorem 2. Because of (0'), \( T \) is symmetric (resp. self-adjoint). The necessity of (0'), (1') and (2') is obvious.

**REFERENCES**

