ON THE ASYMPTOTIC BEHAVIOUR OF THE $G_\theta^\kappa$-MEANS OF EIGENFUNCTION EXPANSION RELATED TO THE SLOWLY OSCILLATING FUNCTIONS WITH REMAINDERTERM

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Abstract. Let $f(Q) = f(x_1, \ldots, x_n) \in L^2(D)$ where $D$ is a bounded open domain with the sufficiently regular boundary in the space $E^n$. Two theorems are proved in this paper. The main result is expressed by Theorem 2 which connects the asymptotic behaviour of the $G_\theta^\kappa$ means of eigenfunction expansion (2.1) with the behaviour of the spherical mean of function $f$ when this is related to behaviour of a slowly oscillating function with remainder term.

1. (i) The $G_\theta^\kappa$-method of summation is defined [3] by

\[
G_\theta^\kappa(\lambda, w) = \sum_{\lambda_v \leq w} \left\{ 1 - \exp(\lambda_v - w^\theta) \right\}^{\kappa} a_v
\]

\[ (0 < \lambda_1 < \lambda_2 < \cdots < \lambda_v \to \infty \text{ as } \nu \to \infty), \]

where $0 < \theta < 1$ and $\kappa > 0$, or by

\[
G_\theta^\kappa(w) = \int_0^w \left\{ 1 - \exp(\nu - w^\theta) \right\}^{\kappa} d\nu [A(t)],
\]

where $A(t)$ is of bounded variation in any finite interval. Without loss of generality we can assume $A(0) = 0$ and in this case we have

\[
G_\theta^\kappa(w) = \kappa w^{-\theta} \int_0^w \left\{ 1 - \exp(\nu - w^\theta) \right\}^{\kappa-1} \exp(\nu - w^\theta) A(t) \, dt. \tag{1.1}
\]

(ii) It is quite natural to introduce the class of slowly oscillating functions with remainder term everywhere in analysis whenever results about convergence are extended to more general asymptotic results. They appear naturally in problems related to the asymptotic evaluations of certain integrals and sums.

Definition. Let $r$ be a positive increasing function on $[0, \infty)$ such that

\[
r(x) \to \infty, \ x \to \infty \tag{1.2}
\]

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and

$$x^{-\delta}r(x) \text{ is eventually decreasing}^* \text{ for some } \delta > 0. \quad (1.3)$$

A positive measurable function $L$ on $[0, \infty)$ is called slowly oscillating function with remainder term $r$ if

$$L(tx)[L(x)]^{-1} = 1 + O \left\{ [r(x)]^{-1} \right\}, \quad x \to \infty$$

for every $t > 0$, [1].

We denote the class of these functions by $K_\delta(r)$. We will use the following properties of functions of class $K_\delta(r)$ [1]:

(a) If $\lambda > 0$, then

$$x^\lambda L(x) \to \infty, \quad x \to \infty, \quad (1.4)$$

$$x^{-\lambda} L(x) \to 0, \quad x \to \infty. \quad (1.5)$$

(b) If $\lambda > 0$ and

$$L_1(x) = x^{-\lambda} \sup_{0 \leq t \leq x} [t^\lambda L(t)], \quad (1.6)$$

$$L_2(x) = x^\lambda \sup_{x \leq t < \infty} [t^{-\lambda} L(t)], \quad (1.7)$$

then

$$L_1(x) \cong L(x) \quad \text{and} \quad L_2(x) \cong L(x), \quad x \to \infty, \quad (1.8)$$

i.e. both $L_1(x)$ and $L_2(x)$ are slowly oscillating functions.

(c) From (1.4) we get

$$[L(x)]^{-1} \leq x^{\sigma-\delta}, \quad \sigma > \delta > 0, \quad x \geq M. \quad (1.9)$$

(d) Since the function $x^{-\delta}r(x)$ is decreasing, it follows that

$$w^{-\delta} \leq b^{-\delta}r(b)[r(w)]^{-1} \quad \text{for } w \geq b. \quad (1.10)$$

(e) There are the asymptotic relations

$$[L(w)]^{-1} \int_0^1 g(t)L(tw) \, dt = \int_0^1 g(t) \, dt + O \left\{ [r(w)]^{-1} \right\}, \quad w \to \infty \quad (1.11)$$

and

$$[L(w)]^{-1} \int_1^\infty g(t)L(tw) \, dt = \int_1^\infty g(t) \, dt + O \left\{ [r(w)]^{-1} \right\}, \quad w \to \infty \quad (1.12)$$

The conditions which insure the validity of these results are usually of the form

$$\int_0^1 t^{-\lambda} \left| g(t) \right| \, dt < \infty \quad \text{or} \quad \int_1^\infty t^\lambda \left| g(t) \right| \, dt < \infty, \quad \lambda > 0, \quad (1.13)$$

* A function $f$ on $(0, \infty)$ is eventually decreasing if there exists $x_0 \geq 0$ such that $x_2 \geq x_1 \geq x_0$ implies $f(x_1) \geq f(x_2)$. 
assuming that
\[ t^\sigma L(t) \text{ is bounded on } [0, d] \text{ for some } \sigma > \delta, \]
where \([0, d]\) is any finite interval.

(iii) Let \( D \) denote a bounded open domain in the Euclidean space \( \mathbb{E}^n \) \((n \geq 2)\). Let \( P(x_1^0, \ldots, x_n^0) \) and \( Q(x_1, \ldots, x_n) \), be two points in \( D \). We suppose that the boundary \( B \) of the domain \( D \) is sufficiently regular, so that the eigenvalue problem
\[
\Delta u + \lambda u = 0 \quad \text{in } D,
\]
\[
u = 0 \quad \text{on } B,
\]
\[
\Delta u = \sum_{m=1}^{n} \frac{\partial^2 u}{\partial x_m^2}
\]
possesses an infinite number of positive eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \) as \( m \to \infty \) with the corresponding eigenfunctions \( \varphi_1(Q), \varphi_2(Q), \ldots, \varphi_m(Q), \ldots \). We assume that these eigenfunctions form a complete orthonormal set in the space \( L^2 \).

Let \( f(Q) \in L^2(D) \). We form its eigenfunction expansion
\[
f(Q) \sim \sum a_m \varphi_m(Q), \quad (1.15)
\]
where
\[
a_m = \int_D f(Q) \varphi_m(Q) dV_Q
\]
and \( dV_Q \) denotes the element of volume in \( \mathbb{E}^n \).

\( f(P; t) \) is the spherical mean of the function \( f(Q) \) over a sphere of radius \( t \) with centre at the point \( P \), i.e.,
\[
f(P; t) = 2^{-1} \pi^{-n-1} \Gamma(s + 1) \int_S f(x_1^0 + t\xi_1, \ldots, x_n^0 + t\xi_n) dS_\xi, \quad (1.16)
\]
where \( S \) is the unit sphere \( \xi_1^2 + \cdots + \xi_n^2 = 1 \), \( dS_\xi \) its \((n - 1)\)-dimensional volume element and
\[
s = (n - 2)/2. \quad (1.17)
\]
\( \rho_P \) is the shortest distance between the point \( P \) and the boundary \( B \), and \( \rho \) is a number such that
\[
0 < \rho < \rho_P. \quad (1.18)
\]

(iv) \( J_m(x) \) is the Bessel function of the first kind of order \( m \). The following results hold \([6]\)
\[
J_m(x) = O(x^m), \quad x \to 0 \quad (1.19)
\]
\[
J_m(x) = O(x^{-1/2}), \quad x \to \infty \quad (1.20)
\]
\[
\int_0^\infty t^{\nu-1} J_\nu(t) dt = 2^{\nu-1} \Gamma(\mu/2)[\Gamma(1+\nu-\mu/2)]^{-1} \quad (1.21)
\]
for $0 < \mu < \nu + 3/2$.

We will make use of the known formula [6, p. 46]

$$\frac{d^k}{(zd_x)^k} [x^m J_m (x)] = x^{m-k} J_{m-k}(x). \quad (1.22)$$

(v) Throughout this paper all $M, M_0, M_1, \ldots$ are the positive constants.

2. In this paper we consider a problem concerning the asymptotic behaviour of the $G^0_\delta$ means of eigenfunction expansion (1.15), i.e.,

$$G^0_\delta (P; w) = \sum_{\lambda_m \leq w} \{1 - \exp[(\lambda_m - w)w^{-\delta}] \}^\kappa a_m \varphi_m (P). \quad (2.1)$$

We will formulate and prove a theorem (Theorem 2) which connects the asymptotic behaviour of $G^0_\delta$ means (2.1) with the asymptotic behaviour of the spherical mean $f(P; t)$ of the function $f(Q)$ at the point $P$, defined by (1.16), when this is related to the behaviour of a function $L \in K_\delta(r)$.

T. V. Avadhani has proved for the Riesz means of eigenfunction expansion (1.15) ((12.12) in [2]) that

$$R_k (P; w) = \sum_{\lambda_m \leq w} (1 - w^{-1} \lambda_m)^k a_m \varphi_m (P)$$

$$= F_k (P; w) + o[w^{-(k-s+1/2)/2}], \quad w \to \infty, \quad k > s + 1/2, \quad (2.2)$$

where

$$F_k (P; w) = \frac{1}{k!} \frac{d^k}{dw^k} [\varphi_m (P)]$$

$$= c_k w^{(s-k+1)/2} \int_0^w t^{s-k} J_{k+s+1} (t \sqrt{w}) f (P; t) \, dt, \quad (2.3)$$

with

$$c_k = 2^{k-1} \Gamma (k+1) [\Gamma (s + 1)]^{-1}$$

and $s$ defined by (1.17) and $\rho$ by (1.18). The Riesz method of summation is defined by the integral

$$R_k (w) = \frac{1}{k!} \frac{d^k}{dw^k} [\varphi_m (P)]$$

$$= kw^{-k} \int_0^w (w - t)^{k-1} A(t) \, dt. \quad (2.4)$$

If $k$ is a positive integer, then it follows from (2.4) that

$$A(w) = \frac{1}{k!} \frac{d^k}{dw^k} [\varphi_m (P)]$$

$$= \frac{1}{k!} \frac{d^k}{dw^k} [w^k R_k (w)]. \quad (2.5)$$

To prepare for the proof of Theorem 2 we first prove the following

**Theorem 1.** If

$$f(P; t) \equiv t^{-\alpha} L(t^{-1}), \quad t \to 0, \quad L \in K_\delta(r), \quad (2.6)$$

where

$$s + \sigma + 1/2 < \alpha < 2s + 2 - \sigma, \quad s = (n-2)/2, \quad \sigma > \delta > 0, \quad (2.7)$$
then

\[ F_0(P; w) = c_0(\sqrt{w})^{s+1} \int_0^w t^s J_{s+1}(t\sqrt{w}) f(P; t) \, dt \]

\[ = b_0(\sqrt{w})^a L(\sqrt{w}) + O\left\{ (\sqrt{w})^{2s+2a} L(\sqrt{w})[r(\sqrt{w})]^{-1} \right\}, \quad w \to \infty, \]  

(2.8)

with \( c_0 = 2^{-\alpha}[\Gamma(s + 1)]^{-1} \) and

\[ b_0 = 2^{-\alpha} \Gamma(s + 1 - \alpha/2)[\Gamma(s + 1)\Gamma(1 + \alpha/2)]^{-1}. \]  

(2.9)

**Proof.** We write integral (2.8) in the form

\[ F_0(P; t) = c_0(\sqrt{w})^{s+1} \int_0^w t^s J_{s+1}(t\sqrt{w}) L(t^{-1}) \, dt \]

\[ + c_0(\sqrt{w})^{s+1} \int_0^w t^s J_{s+1}(t\sqrt{w}) [f(P; t) - t^{-a} L(t^{-1})] \, dt \]

\[ = H_1 + H_2. \]  

(2.10)

Furthermore,

\[ H_1 = c_0(\sqrt{w})^a \left( \int_0^\infty - \int_0^{1/\sqrt{w}} \right) t^{s-a} J_{s+1}(t^{-1}) L(t\sqrt{w}) \, dt = H_{11} - H_{12}. \]  

(2.11)

Now we estimate the integral

\[ H_{11} = c_0 u^a \int_0^\infty t^{s-a} J_{s+1}(t^{-1}) L(tu) \, dt, \]

where \( u = \sqrt{w}. \)

According to (2.7), (1.19) and (1.20) the function

\[ g(t) = t^{s-a-2} J_{s+1}(t^{-1}) \]

satisfies the conditions (1.13). Therefore we can apply the relations (1.11) and (1.12) and we get

\[ H_{11} = c_0 u^a L(u) \left( \int_0^\infty t^{s-a-2} J_{s+1}(t^{-1}) \, dt + O \left\{ [r(u)]^{-1} \right\} \right), \quad u \to \infty. \]

In virtue of (1.21) we have

\[ \int_0^\infty t^{s-a-2} J_{s+1}(t^{-1}) \, dt = \int_0^\infty t^{s-a} J_{s+1}(t) \, dt \]

\[ = 2^{s-a} \Gamma(1 + s - \alpha/2)[\Gamma(1 + \alpha/2)]^{-1}, \]

i.e.,

\[ H_{11} = b_0 u^a L(u) + O \left\{ u^a L(u)[r(u)]^{-1} \right\}, \quad u \to \infty, \]  

(2.12)
where \(b_0\) is given by (2.9).

With respect to (1.20) we obtain

\[
|H_{12}| \leq Mu^a L(u) \int_0^{1/\rho u} t^{\rho - s - 3/2} L(tu)[L(u)]^{-1} dt
\]

and further by (1.9) it follows that

\[
|H_{12}| \leq Mu^a L(u)u^{-\delta} \int_0^{1/\rho u} t^{\rho - s - \sigma - 3/2} [(tu)^\sigma L(tu)] dt
\]

\[
\leq Mu^a L(u)u^{-\delta} \sup_{0 \leq v \leq 1/\rho} [v^{\sigma} L(v)]_{0}^{1/\rho u} t^{\rho - s - \sigma - 3/2} dt.
\]

In virtue of (1.10) we have

\[
|H_{12}| \leq Mu^a L(u)[r(u)]^{-1} \rho^{\delta} r(1/\rho) \sup_{0 \leq v \leq 1/\rho} [v^{\sigma} L(v)]_{0}^{1/\rho u} t^{\rho - s - \sigma - 3/2} dt
\]

and by (2.7) and (1.14) we finally get

\[
H_{12} = O \left\{ u^a L(u)[r(u)]^{-1} \right\}.
\] (2.13)

With respect to (2.11)–(2.13) we get

\[
H_1 = b_0 u^a L(u) + O \left\{ u^a L(u)[r(u)]^{-1} \right\}, \quad u \to \infty,
\] (2.14)

where \(b_0\) is given by (2.9).

Now we estimate the integral \(H_2\). By assumption (2.6), \(\rho\) can be chosen so that

\[
|f(P; t) - t^{-a} L(t^{-1})| \leq \varepsilon t^{-a} L(t^{-1}), \quad \text{for} \ 0 \leq t \leq \rho,
\]

whence

\[
|H_2| \leq \varepsilon c_0 u^{a+1} \int_0^\rho t^{\rho - a} |J_{\rho+1}(tu)| L(t^{-1}) dt
\]

\[
= \varepsilon c_0 u^a \int_{1/\rho u}^\infty t^{\rho - a - 2} |J_{\rho+1}(t^{-1})| L(tu) dt.
\]

Since \(\text{Im}(x) = O(x^m)\) on \([0, \infty)\) it follows that

\[
|H_2| \leq M\varepsilon u^{a+\sigma} \int_{1/\rho u}^\infty t^{\rho - 2s - 3+\sigma} [(tu)^{-\sigma} L(tu)] dt
\]

\[
\leq M\varepsilon u^{a+\sigma} \left\{ \sup_{1/\rho u \leq v \leq \infty} [v^{-\sigma} L(v)] \right\} \int_{1/\rho u}^\infty t^{\rho - 2s - 3+\sigma} dt.
\]

Since by (2.7) \(a - 2s - 2 + \sigma < 0\), we have

\[
|H_2| \leq M\varepsilon \rho^{a+2s-2-\sigma} \left\{ \sup_{1/\rho u \leq v \leq \infty} [v^{-\sigma} L(v)] \right\} L(u)^{-1} [u^{2s+2} L(u)],
\]
and further by (1.9)

$$|H_2| \leq M_1 \varepsilon \rho^{2+2s-2-\sigma} \left\{ \sup_{1/\rho \leq u < \infty} [u^{-\sigma} L(v)] \right\} u^{-\delta} [u^{2+2s-\sigma} L(u)].$$

With respect to (1.10) we have

$$|H_2| \leq M_1 \varepsilon \rho^{2s+2-2s+\delta-\sigma} r(1/\rho) \left\{ \sup_{1/\rho \leq u < \infty} [u^{-\sigma} L(v)] \right\} u^{2s+2-\sigma} L(u) [r(u)]^{-1}.$$

According to the property (1.5), $\rho$ can be chosen so that

$$H_2 = O \left\{ u^{2s+2+\sigma} L(u) [r(u)]^{-1} \right\}, \quad u \to \infty. \quad (2.15)$$

In virtue of (2.10), (2.14), (2.15), (2.7) and substituting $u = \sqrt{w}$ we finally get

$$F_0(P; w) = b_0 (\sqrt{w})^\sigma L(\sqrt{w}) + O \left\{ (\sqrt{w})^{2s+2+\sigma} L(\sqrt{w}) [r(\sqrt{w})]^{-1} \right\}, \quad w \to \infty, \quad (2.16)$$

where $b_0$ is given by (2.9). This concludes the proof of Theorem 1.

3. Now we formulate and prove the mentioned theorem on the asymptotic behaviour of $G_0^\kappa$-means (2.1) of eigenfunction expansion (1.15)

**Theorem 2.** If

$$f(P; t) \equiv t^{-\alpha} L(t^{-1}), \quad t \to 0, \quad L \in K_\delta(r) \quad (3.1)$$

with

$$s - k + \sigma + 1/2 < \alpha < 2s + 2 - \sigma, \quad k > s + 1/2, \quad s = (n - 2)/2, \quad (3.2)$$

where $\sigma$ is any number such that $\sigma > \delta > 0$, then

$$G_0^\kappa(P; w) = O [w^{1-\theta+\alpha/2} L(\sqrt{w})] + O \left\{ w^{2s+2-\alpha+\sigma/2} L(\sqrt{w}) [r(\sqrt{w})]^{-1} \right\} + o(1), \quad w \to \infty \quad (3.3)$$

for $\kappa > k$, where $k$ is the smallest positive integer greater than $s + 1/2$ and $\theta$ is such that $1/2 < \theta < 1$ and $2^{-1}(2s + 1)(2\theta - 1)^{-1}$ is an integer.

**Proof.** If $k$ is a positive integer, then according to (2.5), (1.1) and (2.2) we can write expression (2.1) in the form

$$G_0^\kappa(P; w) = \kappa w^{-\alpha} \int_0^w \left\{ 1 - \exp[(t-w)^{-\theta}] \right\}^{n-1} \exp[(t-w)^{-\theta}]$$

$$\times \frac{1}{k! \, dt^k} t^k R_k(P; t) dt$$

$$= \kappa w^{-\alpha} \int_0^w \left\{ 1 - \exp[(t-w)^{-\theta}] \right\}^{n-1} \exp[(t-w)^{-\theta}]$$

$$\times \frac{1}{k! \, dt} t^k R_k(P; t) + t^{k-\beta} \varepsilon(t)] dt$$

$$= I_1 + I_2, \quad (3.4)$$

where...
where $\beta = (k - s - 1/2)/2$ and $\varepsilon(t) \to 0$ as $t \to 0$.

In [4, Chapter II] it is proved that

$$I_2 = \frac{(-1)^k}{k!} \int_0^w \frac{d^k}{dt^k} \left( \{1 - \exp[(t - w)w^{-\theta}]\}^{(s-1)} \exp[(t - w)w^{-\theta}] \right) \times t^{k-\beta} \varepsilon(t) \, dt = o(1), \quad w \to \infty,$$  

(3.5)

if $\beta = k(1 - \theta)$ i.e., $(k - s - 1/2)/2 = k(1 - \theta)$, whence

$$k = \frac{2s + 1}{2(2\theta - 1)}; \quad \theta = \frac{1}{2} \left( 1 + \frac{s + 1/2}{k} \right),$$

i.e., $1/2 < \theta < 1$, but such $\theta$ that $2^{-1}(2s + 1)(2\theta - 1)^{-1}$ is an integer, because $k$ is a positive integer.

Now we estimate the integral $I_1$. With respect to (2.3) we have

$$\frac{d^k}{dt^k} \left[ (u\sqrt{t})^{s+1} J_{k+s+1}(u\sqrt{t}) \right] = 2^{-k}u^{2k} \left( u\sqrt{t} \right)^{s+1} J_{s+1}(u\sqrt{t})$$

i.e.,

$$\frac{1}{k!} \frac{d^k}{dt^k} \left[ t^k F_k(P; t) \right] = \frac{(\sqrt{t})^{s+1}}{2\pi^{s+1}} \int_0^\rho u^s J_{s+1}(u\sqrt{t}) f(P; u) \, du = F_0(P; t),$$

and according to (2.8),

$$I_1 = \kappa w^{-\theta} \int_0^w \{1 - \exp[(t - w)w^{-\theta}]\}^{(s-1)} \exp[(t - w)w^{-\theta}] \times \left( b_0(\sqrt{t})^s L(\sqrt{t}) + O \left\{ (\sqrt{t})^{2s+2+\sigma} L(\sqrt{t})[r(\sqrt{t})^{-1}] \right\} \right) \, dt$$

$$= I_{11}(w) + O[I_{12}(w)], \quad w \to \infty.$$  

(3.6)

Since $\kappa > k$ and $k \geq 1$, $k$ is a positive integer, the function

$$T(t, w) = \{1 - \exp[(t - w)w^{-\theta}]\}^{(s-1)} \exp[(t - w)w^{-\theta}], \quad 0 \leq t \leq w$$

has the maximum $\kappa^{-1}(1 - \kappa^{-1})^{s-1}$ for $t = w - w^\theta \log \kappa$.

Therefore

$$I_{11}(w) \leq b_0(1 - \kappa)^{s-1} w^{-\theta} \int_0^w (\sqrt{t})^{s-\sigma} L(\sqrt{t}) \, dt$$

$$\leq b_0(1 - \kappa)^{s-1} w^{-\theta} \int_0^w (\sqrt{t})^{s-\sigma} \sup_{0 \leq \varepsilon \leq \sqrt{t}} \left[ \varepsilon^\sigma L(\varepsilon) \right] \, dt.$$
According to (3.2) it follows that
\[
I_1(w) \leq M w^{-\theta} (\sqrt{w})^{\sigma} \left\{ (\sqrt{w})^{-\sigma} \sup_{0 \leq v \leq \sqrt{w}} [v^{\sigma} L(v)] \right\} w^{1+\alpha/2-\sigma/2}
\]
and with respect to (1.6) and (1.8) we get
\[
I_1(w) = \mathcal{O}[w^{1-\theta+\alpha/2} L(\sqrt{w})], \quad w \to \infty. \tag{3.7}
\]
Finally we estimate the integral \( I_2(w) \).
\[
I_2(w) \leq (1-\kappa^{-1})^{\kappa^{-1}} w^{-\theta} \int_0^w (\sqrt{t})^{2\alpha+2-\delta} \frac{r(\sqrt{t})}{\sqrt{t}} \left\{ (\sqrt{t})^{-\sigma} \sup_{0 \leq v \leq \sqrt{t}} [v^{\sigma} L(v)] \right\} dt
\]
\[
\leq (1-\kappa^{-1})^{\kappa^{-1}} w^{-\theta} \int_0^w (\sqrt{t})^{2\alpha+2-\delta} \frac{r(\sqrt{t})}{\sqrt{t}} \left\{ (\sqrt{t})^{-\sigma} \sup_{0 \leq v \leq \sqrt{t}} [v^{\sigma} L(v)] \right\} \]
\[
\times \int_0^w (\sqrt{t})^{2\alpha+2-\delta} \left[ (\sqrt{t})^{-\delta} r(\sqrt{t}) \right]^{-1} dt.
\]
According to the property (1.3) of the function \( x^{-\delta} r(x) \) we see that the function \([x^{-\delta} r(x)]^{-1} \) is increasing on \([0, \infty)\), and further with respect to (1.6) we get
\[
I_2 \leq (1-\kappa^{-1})^{\kappa^{-1}} w^{-\theta} (\sqrt{w})^{\delta} L_1(\sqrt{w}) \left\{ (\sqrt{w})^{-\delta} r(\sqrt{w}) \right\}^{-1} \int_0^w (\sqrt{t})^{2\alpha+2-\delta} dt
\]
In virtue of (1.8) and 3.2 we obtain
\[
I_2(w) = \mathcal{O}\left\{ w^{2+\alpha/2} L(\sqrt{w}) r(\sqrt{w}) \right\}^{-1} \left\{ w^{-\theta+\alpha/2} L(\sqrt{w}) \right\}, \quad w \to \infty. \tag{3.8}
\]
According to (3.6)–(3.8) we have
\[
I_1 = \mathcal{O}[w^{1-\theta+\alpha/2} L(\sqrt{w})] + \mathcal{O}\left\{ w^{2+\alpha/2} L(\sqrt{w}) \right\}, \quad w \to \infty. \tag{3.9}
\]
Finally from (3.4), (3.5) and (3.9) substituting \( u = \sqrt{w} \) we get (3.3). This concludes the proof of Theorem 2.

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