ON WEAK CONVERGENCE TO THE FIXED POINT OF
A GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAP

R. N. Mukherjee, Tanmoy Som* and Vandana Verma

Abstract. Opial's type of convergence theorem [3] is extended to the case of a
generalized asymptotically nonexpansive map in uniformly convex Banach space having a weak duality
mapping. Bose's result would follow as a corollary to Theorem 3.1 of present work.

1. Introduction. Bose [1] gave a result on asymptotically nonexpansive and
asymptotically regular map which in fact extended Opial's convergence theorem
[3]. We give another generalization of Opial's result by introducing a new type
of generalized asymptotically nonexpansive mapping. Suppose $K$ is a nonempty
closed bounded subset of a Banach space $X$. A mapping $T : K \rightarrow K$ is called
asymptotically nonexpansive (see[1]) if for each $x, y \in K$,

(*) $\|T^i x - T^j y\| \leq k_i \|x - y\|, \quad i = 1, 2, 3, \ldots$

where $\{k_i\}$ is a fixed sequence of positive reals such that $k_i \rightarrow 1$ as $i \rightarrow \infty$.
Existence of fixed points of such a mapping, when $X$ is uniformly convex has been
proved by Goebel and Kirk [2]. In Section 2 we recall some basic definitions and
introduce generalized asymptotically nonexpansive and generalized asymptotically
regular mapping. Also we recall the definition given by Kirk on asymptotically
central set of a sequence. Some results on such a sequence are stated without
proof. Our main results are given in Section 3.

2. Definition. A mapping $T : K \rightarrow K$ is called generalized asymptotically
nonexpansive if,

(2.1) $\|x_i - y_i\| \leq k_i \|x - y\|$

for $x, y \in K$, where $x_i$ is defined by Mann-type iterations, and

$x_i = \lambda x_{i-1} + (1 - \lambda)T x_{i-1}, \quad i = 1, 2, 3, \ldots, \lambda < 1$

*Department of Mathematics, P.U. College, Aizawi, Mizoram.

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where \( x_0 = x \), \( \{k_i\} \) is a sequence of real numbers such that \( k_i \to 1 \) as \( i \to \infty \). \( T \) is \textit{generalized asymptotically regular} if for any \( x = x_0 \) in \( k \),

\[
x_i - x_{i+1} \to 0 \quad \text{as} \quad t \to \infty.
\]

The mapping \( T \) is said to be demiclosed if for any sequence \( x_n \in K, x_n \to x_0 \) (weakly), \( Tx_n \to y_0 \Rightarrow Tx_0 = x_0 \). The modulus of convexity of \( X \) is a function \( \delta : [0, 2] \to [0, 1] \) defined by

\[
\delta(\varepsilon) = \inf \{1 - \|x + y\|/2 : \|x\| \leq 1, \|x - y\| \geq \varepsilon\}.
\]

It is known that \( \delta \) is a nondecreasing function and continuous on \([0, 2]\). It is also known that

\[
\|x\| \leq \delta, \quad \|y\| \leq \delta,
\]

(\(\star\star\))

\[
x - y \| \geq \varepsilon \Rightarrow \|x + y\| \leq (1 - \delta(\varepsilon/\delta))d
\]

Opič [3] has shown that in an uniformly convex Banach space having weakly continuous duality mapping (or in a Hilbert space) if a sequence \( \{x_n\} \) converges weakly to \( x_0 \) then

\[
\lim \inf \|x_n - x_0\| < \lim \|x_n - x\|, \quad \forall x \neq x_0.
\]

\textit{Remark.} Observe that the definitions (\(\star\)) and (2.1) are independent of each other.

3. Let \( K \) be a nonempty bounded closed convex subset of a reflexive Banach space \( X \) and let \( \{x_n\} \) be any sequence in \( K \). Following Kirk and Edelstein (see [1]) we define

\[
r(x) = \lim \sup \|x_n - x\|, \quad x \in X.
\]

This \( r \) is a continuous function of \( X \) into reals [1].

Let \( \rho = \rho(\{x_n\}) = \inf \{r(x) : x \in K\} \) and \( C_0 = \{x \in K : r(x) = \rho\} \). \( \rho \) is called the asymptotic radius of \( \{x_n\} \) in \( K \) and \( C_0 \) is the asymptotically central set of \( \{x_n\} \) in \( K \). \( C_0 \) is a singleton if \( X \) is uniformly convex. In that case it is called the asymptotic center.

Let \( B_n(r) \) denote the closed ball of radius \( r \) centered at \( x_n \) and define

\[
C_\varepsilon = \bigcup_{j \geq 1} (\cap_{n \geq j} B_n(\rho + \varepsilon))
\]

\textbf{Proposition 3.1.} \( C_0 = \cap_{\varepsilon > 0} (K \cap C_\varepsilon) \) and is a nonempty closed convex subset of \( K \).

\textbf{Proposition 3.2.} If the space is uniformly convex then \( C_0 \) is a singleton.

As a consequence of Proposition 3.2 we derive the following lemma.
Lemma 3.1. Let $K$ be a nonempty bounded closed convex subset of a uniformly convex Banach space having weakly continuous duality mapping. If a sequence \( \{x_n\} \subset K \) converges weakly to a point $x_0$ then $x_0$ is the asymptotic centre of \( \{x_n\} \) in $K$.

Lemma 3.2. Let $K$ and $X$ be as in Lemma 3.1 and let $T : K \to K$ be a generalized asymptotically nonexpansive mapping. Suppose $x_0$ is the asymptotic centre of the sequence \( \{x_n\} \) for some $x$ in $K$. If the weak limit $\varepsilon_0$ of the subsequence \( \{x_{n_i}\} \) is a fixed point of $T$, then it must coincide with $x_0$.

Proof. Let $\rho$ and $\rho'$ be the asymptotic radii respectively of \( \{x_n\} \) and \( \{x_{n_i}\} \). Clearly $\rho' \leq \rho$. Since \( \{x_{n_i}\} \) converges weakly to $\xi_0$, by lemma 1, $\xi_0$ must ne the asymptotic centre of \( \{x_{n_i}\} \) in $K$, so given $\varepsilon > 0$, we can choose an integer $i_0$ such that $\|\xi_0 - \{x_{n_{i_0}}\}\| \leq \rho' + \varepsilon/2$. Since $\xi_0$ is a fixed point of $T$, we get $\xi_{0_{i_0}} = \xi_0$, and since $T$ is generalized asymptotically nonexpansive, we can choose an integer $J$ such that,

\[
\|\xi_0 - x_{n_{i_0} + j}\| = \|\xi_{0_j} - x_{n_{i_0} + j}\| \leq k_j\|\xi_0 - x_{n_{i_0}}\| \\
\leq k_j(\rho' + \varepsilon/2) \leq \rho' + \varepsilon \leq \rho + \varepsilon, \text{ for all } j \geq J.
\]

It follows therefore that $\lim_{n} \sup \|\xi_0 - x_n\| = \rho$ and, $x_0$ being the unique point with this property, we have $x_0 = \xi_0$.

Our main convergence theorem goes as follows.

Theorem 3.1. Let $X$ be a uniformly convex Banach space having weakly continuous duality mapping and $K$ a nonempty closed bounded convex subset of $X$. Suppose $T$ is a continuous generalized asymptotically nonexpansive mapping, and generalized asymptotically regular. Then for any $x$ in $K$, the sequence \( \{x_n\} \) converges weakly to a point of $T$.

Proof. We will show that the generalized asymptotic regularity of $T$ makes every weak cluster point of \( \{x_n\} \) a fixed point of $T$. In view of Lemma 3.1 this would mean that all the weak cluster points of \( \{x_n\} \) coincide with the asymptotic centre $x_0$ of \( \{x_n\} \) in $K$ (which is fixed point) and would complete the proof.

Let us suppose that the subsequence \( \{x_{n_i}\} \) converges weakly to $\xi_0$. Then, by Lemma 3.1, $\xi_0$ is the asymptotic centre of \( \{x_{n_i}\} \) in $K$. Let the asymptotic radius be $\rho$. By generalized asymptotic regularity of $T$,

\[
x_{n_{i+1}} - x_{n_i} \to 0 \text{ as } i \to \infty.
\]

Since \( \{x_{n_i}\} \) converges weakly to $\xi_0$, this implies \( \{x_{n_{i+1}}\} \) converges weakly to $\xi_0$.

It follows in the same way that for any integer $r$, \( \{x_{n_{i+r}}\} \) converges weakly to $\xi_0$. Thus all these sequence have the same asymptotic centre $\xi_0$ in $K$. We now claim that all these sequences have the same asymptotic radius $\rho$.

We have

\[
\|\xi_0 - x_{n_{i+1}}\| - \|\xi_0 - x_{n_i}\| \leq \|\xi_0 - x_{n_{i+1}}\| - (\xi_0 - x_{n_i}) \|
\]

\[
\leq \|x_{n_{i+1}} - x_{n_i}\| \to 0 \text{ as } i \to \infty.
\]
by generalized asymptotic regularity of $T$. Thus
\[
\limsup_i \|\xi_0 - x_{n_{i+1}}\| = \limsup_i \|\xi_0 - x_{n_i}\| = \rho
\]
and our claim follows.

We now prove that $\xi_0$ is a fixed point of $T$. For this it suffices to show that $\xi_{0j} \to \xi_0$ as $j \to \infty$. Indeed
\[
(1 - \lambda)\|T \xi_{0j} - \xi_0\| = \|(1 - \lambda)T \xi_{0j} - (1 - \lambda)\xi_0\|
\]
\[
= \|\xi_{0j+1} - \lambda \xi_{0j} - (1 - \lambda)\xi_0\| \to 0
\]
as $j \to \infty$, since $\xi_{0j} \to \xi_0$ as $j \to \infty$. Thus $T \xi_{0j} \to \xi_0$ as $j \to \infty$ and since $T$ is continuous, it follows that $\xi_0$ is a fixed point of $T$.

Let us suppose now that $\xi_{0j}$ does not converge to $\xi_0$. Then there is a $d > 0$ and a sequence $\{j_m\}$ of integers such that
\[
\|\xi_0 - \xi_{0j}\| \geq 0 \text{ for all } m.
\]

By uniform convexity of the space, we may choose an $\varepsilon > 0$ such that
\[
(\rho + \varepsilon)[1 - \delta(d/(\rho + \varepsilon))] < \rho.
\]
Since all the sequences $\{x_{n_{i+r}}\}_{r=0}^\infty$ have the same asymptotic centre $\xi_0$ and same asymptotic radius $\rho$, there exist integers $I = I(r)$ such that
\[
\|\xi_0 - x_{n_{i+r}}\| \leq \rho + \varepsilon \text{ for all } i \geq I(r).
\]

We have for any $m$
\[
\|\xi_{0j_m} - x_{n_{i+j_m}}\| \leq k_{j_m} \|\xi_0 - x_{n_i}\| \leq k_{j_m} (\rho + \varepsilon/2) \text{ for } i \geq I(0),
\]

We choose an integer $M$ such that (as $k_j \to 1$ as $j \to \infty)k_{j_m} (\rho + \varepsilon/2) \leq \rho + \varepsilon$ for all $m \geq M$, so that we have
\[
\|\xi_{0j_m} - x_{n_{i+j_m}}\| \leq \rho + \varepsilon \text{ for all } i \geq I(0) \text{ and all } m \geq M
\]
and from (1) we have
\[
\|\xi_0 - x_{n_{i+j_m}}\| \leq \rho + \varepsilon \text{ for all } i \geq I(j_m),
\]

since $\|\xi_0 - \xi_{0j_m}\| \geq d$, (3) and (4) yield
\[
\|(\xi_0 - \xi_{0j_m})/2 - x_{n_{i+j_m}}\| \leq (\rho + \varepsilon)[1 - \delta(d/\rho + \varepsilon)] < \rho
\]
for all $i \geq \max\{I(0), I(j_m)\}$. This contradicts the fact that the sequence $\{x_{n_{i+j_m}}\}_{i=1}^\infty$ has asymptotic radius $\rho$ in $K$ and so completes the proof.
Remark 1. The existence proof for a fixed point of a continuous generalized asymptotically nonexpansive mapping can be given in the same fashion as in the case of an asymptotically nonexpansive mapping (see Joshi and Bose [2, Theorem 4.2.20, p. 111]).

Remark 2. Theorem 3.1 implies the corresponding result of Bose [1] by taking \( \lambda = 0 \) in the definition of generalized asymptotic nonexpansiveness given at the beginning of Section 2.

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Department of Applied Mathematics
Institute of Technology
Banaras Hindu University
Varanasi, 221005, India

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