ON CLOSE-TO-CONVEX FUNCTIONS

D. K. Thomas

Abstract. Well-known coefficient and length results for the class of univalent close-to-
convex functions are extended to a subclass of close-to-convex functions of high order.

1. Introduction. In [3] Goodman introduced the class $K(\beta)$ of normalised
analytic functions which are close-to-convex of order $\beta \geq 0$, i.e. $f \in K(\beta)$ if $f$
is analytic in $D = \{ z : | z | < 1 \}$ and if there exists $\varphi \in K(0) = C$ the class of
normalised convex functions, such that for $z \in D$,

$$\arg \frac{f'(z)}{\varphi'(z)} \leq \frac{\beta \pi}{2}. $$

When $0 \leq \beta \leq 1$, $K(\beta)$ consists of univalent functions, whilst if $\beta > 1$ $f$ need not
even be finitely valent.

Denote by $V_k, (k \geq 2)$ the class of locally univalent functions with bounded
boundary rotation and by $K_k$ the class of functions with bounded radial rotation.
Then $\varphi \in V_k$ if, and only if, $\varphi' \in K_k$ (see e.g. [2]). In [5] Noor considered the class
$T_k$ defined as follows:

Definition. Let $f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n$ be analytic and locally univalent in
$D$. Then for $k \geq 2$, $f \in T_k$ if there is a function $\varphi \in V_k$ such that for $z \in D$,

$$\Re \frac{f'(z)}{\varphi'(z)} > 0$$

Clearly $T_k = K(1)$, the class of close-to-convex functions and it is easily seen [5]
that $T_k \subset K(k/2)$ for $k \geq 2$.

For $f \in K(1)$, Clunie and Pommerenke [1] showed that for $n \geq 2$, $n \mid \alpha_n \mid <
(2 + \sqrt{2})e M(n/(n+1))$, where $M(r) = \max_{0 \leq \theta \leq \pi} | f(re^{i\theta}) |$ and the author [7] showed
that $L(r) < A M(r) \log 1/(1 - r)$, where $L(r)$ denotes the length of the image of
$\{ z : | z | = r \}$ by $f(z)$ and where $A$ is an absolute constant. The object of the

AMS Subject Classification (1980 revision): Primary 30 C 45
present paper is to extend these results to the class $T_k$. The question of whether
the results remain valid in the wider class $K(\beta)$ for $\beta > 1$ remains open.

2. Results. **Theorem 1.** Let $f \in T_k (k \geq 2)$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then for $n \geq 2$,
$$n \mid a_n \mid \leq 3k e M(n/(n+1))$$
(2)

Proof. We modify the method of Clunie and Pommerenke [1]. From (1) write
$$zf'(z) = g(z)\overline{h(z)},$$
(3)
so that $g \in R_k$, $h(0) = 1$ and $\Re h(z) > 0$ for $z \in D$.
Thus we can write $zf'(z) = 2g(z)\Re h(z) - g(z)\overline{h(z)}$. Now with $z = re^{i\theta}$,
$$n a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z)e^{-in\theta} d\theta$$
$$= \frac{1}{\pi r^n} \int_0^{2\pi} g(z)\Re [h(z)]e^{-in\theta} d\theta - \frac{1}{2\pi r^n} \int_0^{2\pi} g(z)\overline{h(z)}e^{-in\theta} d\theta.$$ 
Therefore
$$n \mid a_n \mid \leq \frac{1}{\pi r^n} \int_0^{2\pi} \mid g(z) \mid \Re [h(z)] d\theta + \frac{1}{2\pi r^n} \int_0^{2\pi} \mid g(z)\overline{h(z)}e^{-in\theta} d\theta.$$ 
Since $\Re h(z) > 0$ for $z \in D$, (3) gives
$$\mid g(z) \mid \Re [h(z)] = \Re [zf'(z)e^{-i\arg g(z)}].$$ 
Thus integrating by parts
$$I_1(r) = \frac{1}{\pi r^n} \Re \int_0^{2\pi} f(z)e^{-1\arg g(z)} d\theta (\arg g(z)) \leq \frac{k}{r^n} M(r),$$
since
$$\int_0^{2\pi} \mid \Re \frac{zf'(z)}{g(z)} \mid d\theta \leq k\pi$$
(4)
For $I_2(r)$, we have from (3)
$$I_2(r) = \frac{1}{2\pi r^{2n}} \int_0^{2\pi} z^{n+1} f'(z)e^{-2i\arg g(z)} d\theta.$$ 
(5)
Let $f_n(z) = \int_0^z t^n f'(t) dt$. Then integrating by parts gives
$$\mid f_n(z) \mid \leq 2r^n M(r).$$
(6)
Finally integrating by parts in (5) shows that
$$I_2(r) = \frac{1}{\pi r^{2n}} \int_0^{2\pi} f_n(z)e^{-2i\arg g(z)} \Re \frac{zf'(z)}{g(z)} d\theta \leq \frac{2k}{r^n} M(r)$$
on using (4) and (6).

Choosing \( r = n/(n + 1) \) gives (2).

**Theorem 2.** Let \( f \in T_k(k \geq 2) \). Then for \( 0 < r < 1 \),

\[
L(r) \leq A(k)M(r) \log 1/(1 - r),
\]

where \( A(k) \) is a constant depending only upon \( k \).

**Proof.** With \( z = re^{i\theta} \), (3) gives

\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \leq \int_0^{2\pi} \int_0^r \left| g'(re^{i\theta})h(re^{i\theta}) \right| \, d\theta \, dp + \int_0^r \int_0^{2\pi} \left| g(re^{i\theta})h'(re^{i\theta}) \right| \, d\theta \, dp = J_1(r) + J_2(r)
\]

Now \( J_1(r) = \int_0^r \int_0^{2\pi} |f'(re^{i\theta})H(re^{i\theta})| \, d\theta \, dp \), where \( H(z) = \frac{zg'(z)}{g(z)} \). Thus

\[
J_1(r) \leq 2\pi \int_0^r \left( \int_0^{2\pi} \left| f'(re^{i\theta})^2 \right| \, d\theta \right) \frac{1}{r^2} \left( \frac{r^2}{1 - \rho^2} \right) \left( 1 + \frac{(k^2 - 1)\rho^2}{1 - \rho^2} \right)^{1/2} \, d\rho
\]

(7)

where we have used the Cauchy-Schwartz inequality, Parseval’s equality and Lemma 2 in [5].

If \( f \in K(\beta) \), \( 0 \leq \beta \leq 1 \), then \( f \) is univalent in \( D \) [3]. However for \( \beta > 1 \), \( f \) need not be finitely valent [4]. Thus to estimate the first expression in (7) we proceed as follows.

With \( \rho = n/(n + 1) \), (2) gives

\[
\sum_{n=2}^{\infty} n^2 |a_n|^2 \rho^{2n-2} \leq 9k^2e^2 M(\sqrt{\rho})^2 \sum_{n=2}^{\infty} n^{n-2},
\]

(8)

It follows immediately from the definition of \( T_k \) that the class \( T_k \) forms a subset of a linear-invariant family of order \( k/2 + 1 \). Using Lemma 2.6 of [6] we deduce that \( M(\sqrt{\rho}) < 2^{k+2}M(\rho)/\sqrt{\rho} \). Thus from (7) and (8) we have \( J_1(r) < A(k)M(r) \log 1/(1 - r) \).

To estimate \( J_2(r) \) we note that since \( \Re h(z) > 0 \) for \( z \in D \), \( |h'(re^{i\theta})| \leq 2\Re h(re^{i\theta})/(1 - \rho^2) \). Thus

\[
J_2(r) \leq 2 \int_0^r \int_0^{2\pi} \frac{g(re^{i\theta})}{1 - \rho^2} \frac{\Re h(re^{i\theta})}{1 - \rho^2} \, d\theta \, dp \leq 2k\pi \int_0^r \frac{M(\rho)}{1 - \rho^2} \, dp
\]

as in the proof of Theorem 1. Combining the estimates for \( J_1(r) \) and \( J_2(r) \) gives Theorem 2.
Remark. The proof of Theorem 2 shows that in fact  

\[ L(r) \leq A(k) \int_0^r \frac{M(\rho)}{1-\rho} d\rho. \]

Thus if \( f \in T_k \) and \( M(r) < 1/(1-r)^\alpha \), \( \alpha > 0 \), then \( L(r) < A(k, \alpha)/(1-r)^\alpha \), where \( A(k, \alpha) \) denotes a constant depending only upon \( k \) and \( \alpha \).

REFERENCES