FREDHOLM THEORY AND SEMILINEAR EQUATIONS WITHOUT RESONANCE INVOLVING NONCOMPACT PERTURBATIONS, I.

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Dedicated to Academician Duško Kupera, on the occasion of his eightieth birthday, in gratitude.

1. Introduction. Nonlinear Fredholm theory began with the works of Lasota [9] and Lasota-Opial [10] for (multivalued) compact maps and has attracted the attention of many authors. Since then, extensions of the first Fredholm theorem and of the Fredholm alternative in a weaker form (i.e., without the dimension assertion) have been obtained for various classes of nonlinear maps, like compact, (set) condensing, of types $(S)$ and $(S_+)$, monotone and $A$-proper ones (cf. [3, 4, 5, 6, 18, 19, 23]). In contrast to the works of other authors, in [11-15] we began developing a Fredholm theory for (pseudo) $A$-proper type of maps that are asymptotically close to a suitable map (cf. (2.2)) and, in particular, have a positive quasinorm (cf. (2.2)).

The purpose of this paper is twofold. First, in Section 2, we prove a rather general extension of the first Fredholm theorem for equations of the form

\begin{equation}
Tx = f \quad (x \in X, \ f \in Y)
\end{equation}

where $X$ and $Y$ are normed, linear spaces and $T : X \to Y$ is either (pseudo) $A$-proper or a uniform limit of $A$-proper maps. When $T = A + N$ is pseudo $A$-proper with $A : D(A) \subset X \to Y$ linear and $N$ nonlinear with quasinorm $N \geq 0$, we also prove a weaker form of the Fredholm alternative for semilinear equations

\begin{equation}
Ax + Nx = f \quad (x \in D(A), \ f \in Y).
\end{equation}

In case when $A + N$ is a continuous $A$-proper map, we prove a complete Fredholm alternative (Theorem 2.3). Second, in Section 3, using these results, we study the solvability of Eq. (1.2) with $\text{dimker}(A) \leq \infty$ when there is no resonance at infinity.

AMS Subject Classification (1980): Primary 47H15.
Moreover, the case of nonlinear $A$ is also studied. Due to the generality of the $A$-proper like maps, the obtained results are applicable to many different classes of nonlinear maps mentioned above. We also note that, using a degree theory for multivalued maps, the results of this paper are also valid for multivalued maps $T$ and $N$. Applications of the theory to integral and partial differential equations are given in Part II (this issue).

2. Fredholm theory. Let $\{E_n\}$ and $\{F_n\}$ be sequences of finite dimensional spaces and $\{V_n\}$ and $\{W_n\}$ be sequences of continuous linear maps with $V_n$ mapping $E_n$ into $X$ injectively and $W_n$ mapping $Y$ onto $E_n$. Suppose that $\text{dist}(x, V_n E_n) \to 0$ as $n \to \infty$ for each $x \in X$, dim $X_n = \text{dim } Y_n$ for each $n$ and $\delta = \max \|Q_n\| < \infty$. Then $\Gamma = \{E_n, V_n; F_n, W_n\}$ is said to be an admissible scheme for $(X, Y)$. In particular, let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of $X$ and $Y$ respectively, and $P_n : X \to X_n$ and $Q_n : Y \to Y_n$ be linear projections onto $X_n$ and $Y_n$ with $P_n x \to x$ and $Q_n y \to y$ for each $x \in X$ and $y \in Y$. If $V_n = P_n | X_n = I_n$, then $\Gamma_0 = \{X_n, P_n; Y_n, Q_n\}$ is a projectively complete scheme for $(X, Y)$.

Let $D \subset X$, $T : D \to Y$ and $T_n \equiv W_n T Y_n : D_n = V^{-1}(D) \to F_n$. Recall [21].

**Definition 2.1.** A map $T : D \to Y$ is $A$-proper (pseudo $A$-proper) w.r.t. $\Gamma$ if $T_n$ is continuous for each $n$ and, whenever $\{V_n u_{nk} | u_{nk} \in D_n\}$ is bounded and $\|T_{nk} u_{nk} - W_{nk} f\| \to 0$ as $k \to \infty$ for some $f \in Y$, then some subsequence $V_{nk(i)} u_{nk(i)} \to x$ (there is an $x$, respectively) with $T x = f$.

We say that the equation $T x = f$ is feebly approximation (f. a.) solvable w.r.t. $\Gamma$ if $T_n u_n = W_n f$ for some $u_n \in D_n$, $n \geq 1$, and some subsequence $V_{nk} u_{nk} \to x$ with $T x = f$. The theory of (pseudo) $A$-proper maps is well developed and we refer to, e.g., [14–16, 21–23], where one can find also many examples of such maps.

Our first result is the following generalized first Fredholm theorem.

**Theorem 2.1.** Let $A$, $T : X \to Y$ be nonlinear maps such that

**(2.1)** There are an $n_0 \geq 1$ and a function $c : R^+ \to R^+$ such that $c(r) \to \infty$ as $r \to \infty$ and $\|W_n A x_n\| \geq c(\|x\|)$ for $x \in V_n (E_n)$ and $n \geq n_0$.

**(2.2)** $T$ is asymptotically close to $A$, i.e.

$$|T - A| = \limsup_{\|x\| \to \infty} \frac{\|T x - A x\|}{c(\|x\|)} < 1/\delta.$$

**(2.3)** There is an $R > 0$ such that either $A$ is odd on $X \setminus B(0, R)$ or, for each $r \geq R$, the Brouwer degree $\deg(T_n + \mu G_n, B_n(0, r), 0) \neq 0$ for all large $n$, some bounded map $G : X \to Y$ and all $\mu \in (0, \mu_0)$ with $\mu_0$ small. Then

(a) If $T$ is $A$-proper w.r.t. $\Gamma$ and $\mu = 0$ in (2.3), Eq. (1.1) is f.a. solvable for each $f \in Y$.

(b) If $T + \mu G$ is $A$-proper w.r.t. $\Gamma$ for each $\mu \in (0, \mu_0)$ and $T$ satisfies condition (*) (i.e. whenever $Tx_n \to f$ with $\{x_n\}$ bounded, then $Tx = f$ for some $x$), then $T$ is surjective, i.e. $T(X) = Y$. 

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(c) If $T$ is pseudo $A$-proper w.r.t. $\Gamma$ and $\mu = 0$ in (2.3), then $T(X) = Y$.

Proof. We shall first consider the case when $A$ is odd on $X \setminus B(0, R)$ in (2.3). Then parts (a) and (c) have been proved in [11, 12] and [15], respectively. The validity of part (b) has already been named in [12, 15] (cf. also [14]) without proof and we shall prove it now using a finite dimensional antipodes theorem of Borsuk.

Let $f \in Y$ be fixed. Then, since the map $Bx = Tx - f$ has the same properties as $T$, it suffices to show that $Tx = 0$ is solvable. Let $\varepsilon \geq 0$ be such that $|T - A| + 2\varepsilon < 1/\delta$ and $r \geq R$ such that $c(r) \geq 1$ and $\|Tx - Az\| \leq ((T - A) + \varepsilon) c(|z|)$ for each $|z| \geq r$. Since $G$ is bounded, there is $\mu_1 \in (0, \mu_0)$ such that $\mu_1 \|Gz\| < \varepsilon$ for all $|z| = r$. Then, for each $\mu \in (0, \mu_1)$ and $|z| = r$, we have

$$\|Tx + \alpha Gz - Az\| \leq (|T - A| + 2\varepsilon) c(r) < c(r)/\delta.$$ 

Let $\mu \in (0, \mu_1)$ be fixed. Then, for each $n \geq 1$,

$$(2.4) \quad T_n(u) + \mu G_n(u) \neq \lambda T_n(-u) + \mu G_n(-u) \quad \text{for} \quad u \in \partial B_n(0, r), \; \lambda \in [0, 1].$$

If not, then there would exist an $u_n \in \partial B_n(0, r)$ and $\lambda \in [0, 1]$ such that $(T_n + \mu G_n)(u_n) = \lambda (T_n + \mu G_n)(-u_n)$ for some $n$. Hence,

$$\frac{1}{1 + \lambda} (A_n - T_n - \mu G_n)(u_n) = \frac{\lambda}{1 + \lambda} (T_n + \mu G_n - A_n)(-u_n) = A_n u_n$$

and therefore

$$c(\|V_n u_n\|) \leq \|A_n u_n\| \leq \frac{\delta}{1 + \lambda} \|T + \mu G - A\| V_n u_n\| + \frac{\delta \lambda}{1 + \lambda} \|T + \mu G - A\| (-V_n u_n\|) < c(\|V_n u_n\|),$$

a contradiction. Hence, (2.4) holds and consequently, for each $n \geq 1$ there is an $u_n \in \partial B_n(0, r)$ such that $T_n u_n + \mu G_n u_n = 0$ by the Borsuk antipodes theorem. Since $T + \mu G$ is $A$-proper, a subsequence $V_n u_{n_k} \rightarrow x \in \overline{B}(0, r)$ with $Tx + \mu Gx = 0$. Next, let $\mu_k \rightarrow 0$ and $T x_k + \mu_k G x_k = 0$ for some $x_k \in \overline{B}(0, r)$. Since $G$ is bounded, $T x_k \rightarrow 0$ and $Tx = 0$ for some $x \in X$ by condition (*).

Next, let us suppose in (2.3) that for each $r \geq R$ and $\mu \in [0, \mu_0]$, $\deg (T_n + \mu G_n, B(0, r), 0) \neq 0$ for all large $n$. When $\mu = 0$, this happens if, for example, $T$ is odd on $X \setminus B(0, R)$ or if $(T x, K x) \geq 0$ for $|x| \geq R$ and some additional conditions on $K : X \rightarrow Y$ and $\Gamma$ (cf., e.g., [14, 21]). Part (a) has been proved in [12] in these special cases and, using similar arguments, we shall now give a unified proof of the parts (a)-(c).

Let $f \in Y$ be fixed and define $Bx = Tx - f$, $x \in X$. Then $B$ satisfies (2.2) and let $\beta > 0$ be such that $|B - A| + 2\varepsilon < (1 - \beta)/\delta$. Then there is an $r \geq R$ such that $c(r) \geq \max \{1, 2\delta (|f|/\beta)\}$ and $\|Bx - Az\| \leq (|B - A| + \varepsilon) c(|z|)$ for each $|z| \geq r$. Let $\mu_1 \in (0, \mu_0)$ be such that $\mu_1 \|Bx\| < \varepsilon$ for all $|x| = r$. Then, for each $\mu \in [0, \mu_1)$ and $|x| = r$ we have

$$\|Bx - Az\| \leq \varepsilon + \frac{|B - A| + 2\varepsilon}{1 - \beta} (1 - \beta) c(r)/\delta.$$
Let $\mu \in [0, \mu_1)$ be fixed. Then, for $\|x\| = r$,

\begin{equation}
|W_n(T + \mu G - A)x - tw_nf| \leq |W_n(T + \mu G - A)x - W_nf| + |W_nf|
\end{equation}

\[ \leq \delta (|B - A| + 2\varepsilon) c(r) + c(r)\beta/2 < (1 - \beta/2)c(r). \]

For $B_n = V_{n}^{-1}(B(0,r)) \subset E_n$ we have that $\overline{B} \subset V_{n}^{-1}(\overline{B}(0,r))$ and $\partial B_n \subset V_{n}^{-1}(\partial B(0,r))$. It follows from (2.1) and (2.5) that for each $\mu \in [0, \mu_1)$ fixed, each $u \in \partial B_n$, $n \geq 1$, and $t \in [0,1]$ we have that

\[ ||(T_n + \mu G_n) - tw_nf|| \geq ||A_nu|| - ||(T_n + \mu G_n - A_n)u - tw_nf|| \]

\[ \geq c(||V_n u||) - (1 - \beta/2)c(||V_n u||) = \beta c(||V_n u||)/2 > 0. \]

Hence, for each $\mu \in [0, \mu_1)$ fixed, $(T_n - \mu G_n)u \neq tw_nf$ for $u \in \partial B_n$, $t \in [0,1]$ and $n \geq 1$, and therefore the Brouwer degree $\text{deg}(T_n + \mu G_n, B_n, W_n f) \neq 0$ for each $n \geq 1$.

Now, if $\mu = 0$, it follows that the equation $T_nu = W_nf$ is solvable in $B_n$ for each $n$ and the conclusion of (a) (c), respectively) follows from the $A$-properness (pseudo $A$-properness, respectively) of $T$. In case (b) we have that for each $\mu \in [0, \mu_1)$ fixed the equation $T_nu + \mu G_nu = W_nf$ is solvable in $B_n$ for each $n$, and therefore the equation $T_nu + \mu G_nu = W_nf$ is solvable in $B_n$ for each $n$, and therefore the equation $T_nu + \mu G_nu = W_nf$ is solvable in $B_n$ for each $n$.

The following special cases are useful in applications.

**Corollary 2.1**. Let $T = A + N : X \to Y$, $A$ satisfy (2.1) and

\begin{equation}
|N| = \limsup_{||x|| \to \infty} \frac{||Nx||}{c(||x||)} < 1/\delta.
\end{equation}

Then the conclusions of Theorem 2.1 hold.

**Corollary 2.2**. Let $T = A + N : X \to Y$ with $Q_nAx = Ax$ for $x \in V_nE_n$ and

\begin{equation}
||Ax_n|| \to \infty \quad \text{as} \quad ||x_n|| \to \infty \quad \text{for} \quad x_n \in X;
\end{equation}

\begin{equation}
|N| = \limsup_{||x|| \to \infty} \frac{||Nx||}{||Ax||} < 1/\delta.
\end{equation}

Then the conclusions of Theorem 2.1 hold.

Proof. It follows from Corollary 2.1 by taking $c(||x||) = ||Ax||$ on $X$. □

Regarding condition (2.1), the following lemma is useful [cf. 12, 23].

**Lemma 2.1**. Let $A : X \to Y$ be $A$-proper at $f = 0$ w.r.t. $\Gamma$ and $\alpha$-positively homogeneous (i.e., $A(tx) = t^\alpha Ax$ for $x \in X$, $t > 0$ and some $\alpha > 0$). Then, if $Ax = 0$ implies $x = 0$, there is a constant $c > 0$ and $n_0 > 1$ such that

\begin{equation}
||W_nAz|| \geq c||z||^\alpha \quad \text{for} \quad x \in V_n(E_n), \ n \geq n_0
\end{equation}
Remark 2.1. Theorem 2.1 and Corollaries 2.1–2.2 are applicable to many classes of nonlinear maps and, in particular to (generalized) pseudo monotone ones from $X$ to $X^*$ (cf. [4]). This will be discussed in detail elsewhere.

Next, we shall prove a Fredholm alternative in a weaker form for maps of the form $T = A + N$, where $A$ is a linear Fredholm map of index zero i.e., the kernel $X_0 = N(A)$ and cokernel of $A$ are of the same finite dimension and the range $R(A)$ is closed. We have the direct sums $X = X_0 \oplus \tilde{X}$ and $Y = Y_0 \oplus \tilde{Y}$, $\tilde{Y} = R(A)$, and let $L : X_0 \to Y_0$ be a linear isomorphism and $P : X \to X_0$ be a linear projection onto $X_0$. Then $C = LP : X \to Y_0$ is completely continuous.

**Theorem 2.2.** [17] (Fredholm alternative). Let $A : V \subset X \to Y$ be a linear Fredholm map of index zero with $N(A) \neq \{0\}$ and $A$-proper w.r.t. $\Gamma$ for $(V, Y)$. Let $T : X \to Y$ be nonlinear and such that its range $R(T) \subset R(A)$ and $|T - A| < c/\delta$ for $c$ sufficiently small. Suppose that either

(a) $T$ satisfies condition (*) and $T + \mu G$ is $A$-proper w.r.t. $\Gamma$ for each $\mu \in (0, \mu_0)$ and some bounded map $G : X \to Y$; or

(b) $T + C : V \to Y$ is pseudo $A$-proper w.r.t. $\Gamma$.

Then the equation $Tx = f$ is solvable if and only if $f \in R(A) = N(A^*)^\perp$.

**Proof.** Since $A_1 = A + C$ is injective and $A$-proper w.r.t. $\Gamma$, there is a constant $c > 0$ such that (2.9) holds. Then $T_1 = T + C$ is such that $|T_1 - A_1| < c/\delta$. If (a) holds, then $T_1 + \mu G$ is $A$-proper w.r.t. $\Gamma$ for each $\mu \in (0, \mu_0)$ by the compactness of $C$. In either case, the equation $T_1x = f$ is solvable for each $f \in Y$ by Theorem 2.1. Moreover, if $f \in R(A)$ and $T_1x = f$, then $Cx = f - Tx \in R(A)$ and consequently $Cx = 0$ and $Tx = f$. Conversely, if $Tx = f$ is solvable, then $f \in R(A)$ since $R(T) \subset R(A)$. $\square$

Finally, we shall establish a complete extension of the classical Fredholm alternative for $A$-proper maps of the form $T = A + N$. Recall that the covering dimension of a normal topological space is equal to $n$, provided $n$ is the smallest integer with the property that whenever $U$ is an open covering of $X$, there exist a refinement $U'$ of $U$, which also covers $X$, and no more than $n + 1$ members of $U'$ have nonempty intersection.

**Theorem 2.3.** [17] (Fredholm alternative). Let $A : X \to Y$ be a continuous linear Fredholm map of index zero and $\text{codim}(R(A)) = m > 0$ and $N : X \to Y$ be continuous and such that $|N| < c/\delta$, $R(N) \subset R(A)$ and $T = A + N$ is $A$-proper w.r.t. $\Gamma_0 = \{X_n, P_n; Y_n, Q_n\}$ with $X_0 \subset X_n$ and $Y_0 \subset Y_n$. Then, for each $f \in R(A)(= N(A^*)^\perp)$, and only such ones, there is a connected closed subset $K$ of $T^{-1}(f)$ whose dimension at each point is at least $m$ and the projection $P$ maps $K$ onto $Y_0$.

**Proof.** Let $V_n = Y_n \cap \tilde{Y}$, $X_n = X_0 \oplus U_n$ with $\dim U_n = \dim V_n$ and $\tilde{Q}_n = Q_n[\tilde{Y}]$. Then $T = A + N : X \to Y$ is $A$-proper w.r.t. $\Gamma_m = \{X_n, P_n; V_n, Q_n\}$ with $\dim X_n - \dim V_n = m, n \geq 1$. For a given $f \in R(A)$, let $Bx = Nx - f$. Let $\varepsilon > 0$ be such that $|N| + \varepsilon < c/\delta$ and $R = R(E) > 0$ such that

$$||Nx|| \leq (|N| + \varepsilon)||x||$$

for all $||x|| \geq R$. 


We need to show that $A + B : X_0 \oplus \tilde{X} \rightarrow \tilde{Y}$ is complemented by $P$. To that end it suffices to show (see [2]) that $\deg(Q_n(A + B)|_{U_n}, U_n, 0) \neq 0$ for all large $n$. Define the homotopy $H_n : [0, 1] \times U_n \rightarrow V_n$ by $H_n(t, x_1) = \bar{Q}_n A x_1 + \bar{Q}_n B(x_1)$ We claim that there are $n_0 \geq 1$ and $r \geq R$ such that if, $H_n(t, x_1) = 0$ for some $x_1 \in U_n$ with $n \geq n_0$ and $t \in [0, 1]$ then $||x_1|| < r$. If not, then there would exist $x_{1n_k} \in U_{n_k}$ with $||x_{1n_k}|| \rightarrow \infty$ and $t_k \in [0, 1]$ such that $H_n(t_k, x_{1n_k}) = 0$ for each $k$. Hence,

$$c||x_{1n_k}|| \leq ||\bar{Q}_{n_k} A x_{1n_k}|| \leq \delta(||N|| + \varepsilon)||x_{1n_k}|| + \delta||f||$$

and, dividing by $x_{1n_k}$ and passing to the limit, we arrive at a contradiction to $|N| + \varepsilon < c/\delta$. Thus, the claim is valid and for each $n \geq n_0$, and $\deg(Q_n(A + B)|_{U_n}, U_n, 0) = \deg(Q_n A|_{U_n}, U_n, 0) \neq 0$.

Next, we need to show that $P : X_0 \oplus \tilde{X} \rightarrow X_0$ is proper on $(A + B)^{-1}(0)$. To see this, it suffices to show that if $\{x_n\} \subset X$ is such that $A x_n + B x_n \rightarrow 0$ and $\{P x_n\}$ is bounded, then $\{x_n\}$ is bounded since the $A$-proper map $A + B$ is proper restricted to bounded sets ([21]). We have that $x_n = x_{0n} + x_{1n}$ with $x_{0n} \in X_0$ and $x_{1n} \in \tilde{X}$, and $c||x_{1n}|| \leq ||A x_{1n}|| \leq ||N|| + \varepsilon||x_{1n}|| + ||f||$ for some $\varepsilon > 0$ with $|N| + \varepsilon < c$ if $||x_{1n}|| \geq R$. This implies that $\{x_{1n}\}$ is bounded as before. Since $\{x_{0n}\} = \{P x_n\}$ is bounded, it follows that $\{x_n\}$ is also bounded. Hence, the conclusions of the theorem follow from Theorem 1.2 in Fitzpatrick-Massabó-Pejsachowicz [2]. □

Analogously, a dimension assertion on the solution set of the corresponding “adjoint” equation treated in Theorem 2.3 in [23] can be proven when the involved maps are $A$-proper.

**Remark 2.2.** Theorem 2.2 extends a result of Petryshyn [23] dealing with weakly $A$-proper maps. Moreover, Theorem 2.3 includes the weaker form of the Fredholm alternative (not dealing with the dimension of the solution set) of Kachurovsky [5, 6] for compact maps and of Nečas [18, 19] and Hess [3] for maps of type $(S)$, $(S_+)$ and monotone ones, respectively.

**Remark 2.3.** Using similar arguments, it can be shown that Theorem 2.3 holds for nonlinearities $N$ of superlinear growth, i.e. if $N = N_1 + N_2$ with $N_1$, $A$-proper, odd, $\alpha$-homogeneous for some $\alpha > 1$ and $N_1 x = 0$ implies $x = 0$, and $||N_2 x|| \leq a + b ||x||^k$ for some $a, b, k < \alpha$ and all $x \in X$.

### 3. Applications

We begin by looking at some applications of the abstract results in Section 2 to semilinear equations of the form (1.2) with dimker $A \leq \infty$ when there is no resonance at infinity. By this we mean that there is some linear map $C : V \subset X \rightarrow Y$ such that $0 \notin \sigma(A - C)$, the spectrum of $A - C$, and $N - C$ stays away from $\sigma(A - C)$ at infinity (e.g., (3.1) holds).

Let $H$ denote a real Hilbert space and $X$ and $Y$ be Banach spaces. In the self-adjoint case we have

**Theorem 3.1.** Let $A : D(A) \subset H \rightarrow H$ be self-adjoint, $V = (D(A), ||\cdot||_0)$ be a Banach space density and continuously embedded in $H$, $C : D(C) \subset H \rightarrow H$ be bounded and symmetric with $V \subset D(C)$ and $0 \notin \sigma(A - C)$. Suppose that $N : V \rightarrow H$ is nonlinear and such that
(3.1) There are positive constants $a, b, c, r$ and $k \in (0, 1)$ such that

$$||Nx - Cz|| \leq a||x|| + b||x||^k + c \text{ for } ||x||_0 \geq r$$

(3.2) $0 < a < \min\{\lambda | \lambda \in \sigma(A - C)\}$.

Then, if $A - N : V \to H$ is pseudo $A$-proper w.r.t. $\Gamma_0 = \{X_n, P_n, Y_n, Q_n\}$ for $(V, H)$ with $Q_n(A - C)x = (A - C)x, x \in X_n, n \geq 1$, it is surjective.

Proof. Note first that $B = (A - C)^{-1} : H \to V$ is continuous. Indeed, by the closed graph theorem, it suffices to show that it is closed. Let $x_n \to x$ in $H$ and $Bx_n \to y$ in $V$. Then $Bx_n \to y$ in $H$ and $Bx = y$ by the closedness of $B$ in $H$. Hence, for each $x \in V$

$$||A - C||x|| \geq ||B||^{-1}||x||_0.$$ 

Next, since $C$ is bounded and symmetric, $A - C$ is self-adjoint (see Kato [7, Thm. V. 4.3]) and therefore $\min\{||A|| | \lambda \in \sigma(A - C)\} = ||(A - C)^{-1}||$ and $a||(A - C)^{-1}|| < 1$ by (3.2). Moreover, for each $||x_0|| \geq r$, we have $x = (A - C)^{-1}y$ for some $y \in H$ and

$$||Nx - Cz|| \leq a||(A - C)^{-1}y|| + b||(A - C)^{-1}y||^k + c$$

or

$$\frac{||Nx - Cz||}{||(A - C)||} \leq a||(A - C)^{-1}|| + b||B||^k ||(A - C)x||^{k-1} + c||(A - C)x||^{-1}.$$ 

Hence,

$$|N - C| = \limsup_{||x_0||_0 \to \infty} \frac{||Nx - Cz||}{||(A - C)||} \leq a||(A - C)^{-1}|| < 1$$

and the conclusion follows from Corollary 2.2. □

Remark 3.1. If there are real numbers $\alpha < \beta$ such that a $\sigma(A) \cap (\alpha, \beta)$ consists of at most finitely many eigenvalues, then we can take $C = \lambda I, \lambda = (\lambda_k + \lambda_{k+1})/2$, in Theorem 3.1 for some consecutive eigenvalues $\lambda_k < \lambda_{k+1}$ in $(\alpha, \beta)$. Then (3.2) holds if $a < \gamma = (\lambda_{k+1} - \lambda_k)/2$. Indeed, the spectral gap for $A - \lambda I$ induced by the gap $(\lambda_k, \lambda_{k+1})$ is $(\gamma, \gamma)$ and therefore $(A - \lambda I)^{-1} : H \to H$ is a bounded self adjoint map whose spectrum lies in $(-1/\gamma, 1/\gamma)$. Hence, $||(A - \lambda I)^{-1}|| = 1/\gamma$. Moreover, the scheme $\Gamma_0 = \{(A - \lambda I)^{-1}(Y_n), P_n; Y_n, Q_n\}$ for $(V, H)$ has the required property in Theorem 3.1.

Analyzing the proof of Theorem 3.1, we see that the following more general result holds when $A$ is not selfadjoint.

Theorem 3.2. Let $(V, || \cdot ||_0)$ be density and continuously embedded in $X$, $A : V \to Y$ and $C : X \to Y$ be closed linear maps with $A - C : V \to Y$ bijective. Suppose that $N : V \to Y$ is nonlinear and

(3.3) There are positive constants $a, b$ and $r$, with a sufficiently small such that

$$||Nx - Cz|| \leq a||x||_0 + b \text{ for } ||x|| \geq r.$$
Then, if $A - N : V \to Y$ is pseudo A-proper w.r.t. $\Gamma$ for $(V, Y)$ with $Q_n(A - C)x = (A - C)x$, $x \in X_n$, $n \geq 1$, it is surjective.

Next, we shall look at Eq. (1.2) with nonlinearities of the form $Nx = B(x)x - Mx$, where $B(x) : X \to X$ is a continuous linear map for each $x \in V$ such that for some $\lambda \not\in \sigma(A)$, $A_{\lambda} = A - \lambda I$ and $B_{\lambda}(x) = B(x) - \lambda I$ satisfy

(3.4) \[ m = \limsup_{||x||_0 \to \infty} ||B_{\lambda}(x)|| < \frac{1}{||A_{\lambda}^{-1}||}. \]

**Theorem 3.3.** Let $A : D(A) \subset X \to X$ be a closed linear map, $V = (D(A), || \cdot ||_0)$ be a Banach space densely continuously embedded in $X$ and (3.4) hold. Suppose that $M : V \to X$ is nonlinear and $T : V \to X$, $Tx = A(x) - B(x)x - Mx$, is pseudo A-proper w.r.t. $\Gamma = \{X_n, P_n; Y_n, Q_n\}$. Then

(a) If $Q_n A_{\lambda} x = A_{\lambda} x$, $x \in X$, $n \geq 1$, and there are positive constants $a, b, c, r$ and $k \in (0, 1)$ such that $\delta(a + m) \cdot ||A_{\lambda}^{-1}|| < 1$ and

\[ ||Mx|| \leq a||x|| + b||x||^k_0 + c \quad \text{for} \quad ||x||_0 \geq r, \]

then $T$ is surjective.

(b) If $T_1 x = Ax - B(x)x$ is A-proper w.r.t. $\Gamma_0$ and

\[ |M| = \limsup_{||x||_0 \to \infty} \frac{||Mx||}{||x||_0} < \infty \]

is sufficiently small, then $T$ is surjective.

**Proof.** (a) As in Theorem 3.1, we obtain that

\[ ||A_{\lambda} x|| \geq ||A_{\lambda}^{-1}||^{-1}_{X \to Y} ||x||_0, \quad x \in X. \]

Moreover, for $\varepsilon > 0$ small with $(m + a + \varepsilon)||A_{\lambda}^{-1}|| < 1$ there is an $R > 0$ such that for $||x||_0 \geq R$

\[ ||B_{\lambda}(x)x + Mx|| \leq (m + a + \varepsilon)||x|| + b||x||^k_0 + c. \]

Then, setting $Nx = B(x)x + Mx$ and $C = \lambda I$, the conclusion follows from Corollary 2.2 as in Theorem 3.1.

(b) By (3.4), there is an $R > 0$ such that $||B_{\lambda}(x)|| < 1/||A_{\lambda}^{-1}||$ for all $||x||_0 \geq R$. Hence, for such $x$'s, the map $B_{\lambda}(x)A_{\lambda}^{-1} : X \to X$ satisfies

\[ ||B_{\lambda}(x)A_{\lambda}^{-1}|| \leq ||B_{\lambda}(x)|| ||A_{\lambda}^{-1}|| < \theta < 1 \]

for some $\theta$ independent of $x$. Consequentially, $I - B_{\lambda}(x)A_{\lambda}^{-1} : X \to X$ is invertible and

\[ ||(I - B_{\lambda}(x)A_{\lambda}^{-1})^{-1}|| < 1/(1 - \theta) \quad \text{for} \quad ||x||_0 \geq R. \]

As before, $A_{\lambda}^{-1} : X \to V$ is continuous and therefore $c||x||_0 \leq ||A_{\lambda}x||$ for $x \in V$ and some $c > 0$. Moreover, for $||x||_0 \geq R$

\[ c_1||x||_0 \leq ||[I - B_{\lambda}(x)A_{\lambda}^{-1}][I - B_{\lambda}(x)A_{\lambda}^{-1}]A_{\lambda}x|| \leq ||A_{\lambda}(x) - B_{\lambda}(x)||/(1 - \theta). \]
or

\begin{equation}
(3.5) \quad c_1 \|x\|_0 \leq \|A_\lambda x - B_\lambda(x)x\| \quad \text{for} \quad \|x\|_0 \geq R, c_1 = (1 - \theta)c.
\end{equation}

Since \( T_1 x = A_\lambda x - B_\lambda(x)x = Ax - B(x)x \) is \( A \)-proper, arguing by contradiction and using \((3.5)\), we obtain an \( n_0 \) \geq 1 and \( c_0 \geq 0 \) such that

\begin{equation}
(3.6) \quad c_0 \|x\|_0 \leq \|Q_n(\lambda, B(x))x\| \quad \text{for all} \quad x \in X_n \setminus \overline{B}(0, R), \quad n \geq n_0.
\end{equation}

Since \( |M| \) is sufficiently small, the conclusion follows from Corollary 2.1, where one needs only to assume \((2.1)\) on \( X_n \setminus \overline{B}(0, R) \). \( \square \)

To give some conditions for the \( A \)-properness of \( T_1 \) and \( T \), we recall that a

**ball-measure of noncompactness** of a set \( D \subset X \) is defined by

\[ \chi(D) = \inf \{ r > 0 | D = \bigcup_{i=1}^{n} B(x_i, r), x_i \in X \text{ and some } n \} \].

A map \( T : D \rightarrow Y \) is \( k \)-ball-contractive if \( \chi(T(Q)) \leq k \chi(Q) \) for each \( Q \subset D \). We have

**Proposition 3.1.** Let \( U(x, y) = B(x) y \) for \( (x, y) \in V \times V \) and

\begin{equation}
(3.7) \quad \text{For each } x \in V, \ U(x, \cdot) : V \rightarrow X \text{ is } k_1 \text{-ball-contractive;}
\end{equation}

\begin{equation}
(3.8) \quad \text{For each } y \in V, \ U(\cdot, y) : V \rightarrow X \text{ is completely continuous.}
\end{equation}

Suppose that \( A : V \rightarrow X \) is Fredholm of index zero and \( M : V \rightarrow X \) is \( k \)-ball-contractive with \( k = k_1 + k_2 \) sufficiently small. Then \( T_1, T : V \rightarrow X \) are \( A \)-proper w.r.t. \( \Gamma_0 \) for \( (V, X) \) with \( Q_nAx = Ax \) on \( X_n \).

**Proof.** It is known that the map \( B_1 : V \rightarrow X, B_1(x) = U(x, x) \) is \( k_1 \)-ball-contractive by \((3.7),(3.8)\). Since \( B_1 + M : V \rightarrow X \) is \( k \)-ball-contractive, \( T_1 \) and \( T \) are \( A \)-proper w.r.t. \( \Gamma_0 \) (cf. [15]). \( \square \)

**Remark 3.2.** Condition \((3.7)\) is implied by the compactness of the embedding of \( V \) into \( X \) or by \( \|B(x)\|_{(V \rightarrow X)} \leq k_1 \) for all \( x \in V \). In applications various natural conditions imply \((3.7),(3.8)\).

So far we have studied Eq. \((1.2)\) with nonlinearities \( N \) asymptotically close to linear maps (i.e. when condition of type \((3.1)\) holds). It turns out that when \( A = I \), we can allow more general nonlinearities studied first by Perov [20] and Krasnoselskii-Zabreiko [8]. To introduce this class, we consider a pair of self adjoin maps \( B_1, B_2 : H \rightarrow H \) such that \( B_1 \leq B_2 \), i.e. \( (B_1 x, x) < (B_2 x, x) \) for \( x \in H \), and 1 is not in their spectrum \( \sigma(B_1) \cup \sigma(B_2) \). Let \( \sigma(B_1) \cap (1, \infty) = \{ \lambda_1, \ldots, \lambda_k \} \) and \( \sigma(B_2) \cap (1, \infty) = \{ \mu_1, \ldots, \mu_m \} \), where the \( \lambda_i \)’s and \( \mu_j \)’s are eigenvalues of \( B_1 \) and \( B_2 \), respectively, of finite multiplicities and assume that the sum of the multiplicities of the \( \lambda_i \)’s is equal to the sum of the \( \mu_j \)’s. Then we say that \( B_1 \) and \( B_2 \) form a regular pair.

Recall that \((8)\) a (nonlinear) map \( K : H \rightarrow H \) is said to be \( \{B_1, B_2\}\)-quasilinear on a set \( S \subset H \) if for each \( x \in S \) there exists a linear selfadjoint map \( B : H \rightarrow H \) such that \( B_1 \leq B \leq B_2 \) and \( Bx = Kx \). A map \( N : H \rightarrow H \) is said to be **asymptotically** \( \{B_1, B_2\}\)-quasilinear if there is a \( \{B_1, B_2\}\)-quasilinear outside some ball map \( K \) such that

\begin{equation}
(3.9) \quad |N - K| = \limsup_{\|x\| \to \infty} \frac{\|Nx - Kx\|}{\|x\|} < \infty.
\end{equation}
It has been shown in [8] that if $B_1$ and $B_2$ form a regular pair, then there is a constant $c > 0$ such that for each self-adjoint map $B$ with $B_1 \leq B \leq B_2$ we have that

$$\|x - Bx\| \geq c|x| \quad \text{for each} \quad x \in H.$$  

(3.10)

For example, if $N : H \to H$ is such that $N(x)$ is self-adjoint for each $x$ in $H$ and satisfies

$$B_1 \leq N'(x) < B_2 \quad \text{for} \quad x \in H,$$

then $N$ is asymptotically $\{B_1, B_2\}$-quasilinear since we can represent $Nx = B(x)x + N(0)$, where $B(x) = \int_0^1 N'(tx) dt$. Moreover, if $Nx = B(x)x + Mx$ for some nonlinear $M$ with $|M| < \infty$ and $B(X) : H \to H$ is self-adjoint and $B_1 \leq B(x) \leq B_2$ for each $x$ in $H$, then $N$ is asymptotically $\{B_1, B_2\}$-quasilinear (cf. [20] for some other criteria). For equations with such nonlinearities we have

**Theorem 3.4.** [17]. Let $\{B_1, B_2\}$ form regular pair, $M, N : H \to H$ be bounded and $N$ be asymptotically $\{B_1, B_2\}$-quasilinear with $|M + N - K| < c$. Let $B_0 : H \to H$ be self-adjoint with $B_1 \leq B_0 \leq B_2$ and $H_t = I - t(M + N) - (1 - t)B_0$, $0 \leq t \leq 1$. Then

(a) If $H_t$, is $A$-proper w.r.t. $\Gamma_0 = \{H_n, P_n\}$ for each $t \in [0, 1]$, then the equation $x - Mx - Nx = f$ is f.a. solvable for each $f \in H$.

(b) If $H_t$, is $A$-proper w.r.t. $\Gamma_0$ for each $t < 1$ and $H_1$ is either pseudo $A$-proper w.r.t. $\Gamma_0$ or satisfies condition *(c), then $(I - M - N)(H) = H$.

(c) Let $G : H \to H$ be such that $\|Gx\| < a\|x\|$ on $H$ for some $a$, and for each large $r$, deg$(P_nB_0 + \mu P_nG, B(0, r) \cap X, 0) \neq 0$ for each large $n$ and $\mu > 0$ small. Suppose that $H_1 + \mu G$ is $A$-proper w.r.t. $\Gamma_0$ for each $t \in [0, 1]$ and $\mu > 0$ small and $H_1$ satisfies condition *(a). Then $(I - M - N)(H) = H$.

**Proof.** Since $N_f x = Nx - f$ has the same properties as $N$ for any $t$ in $H$, it suffices to study the equation $x - Mx - Nx = f$. Let $\mu_0 > 0$ and $\varepsilon > 0$ be such that $\|M + N - K\| \varepsilon + a\mu_0 < c$. Then there is an $r > 0$ such that $\|Mx + N x - K x\| \leq (\|M + N - K\| + \varepsilon) \|x\|$ for each $\|x\| \geq r$. Moreover, $H(t, x) + \mu Gx \neq 0$ for $\|x\| = r$, $t \in [0, 1]$ and $\mu \in [0, \mu_0]$. If not, there are $t \in [0, 1]$, $\|x\| = r$ and $\mu \in [0, \mu_0]$ such that $H(t, x) + \mu Gx = 0$. Hence,

$$\|x - tKx - (1 - t)B_0x\| \leq t\|Mx + N x - K x\| + \mu\|Gx\| < c.$$  

Since $K$ is $\{B_1, B_2\}$-quasilinear, there is a self-adjoint map $B_\ast : H \to H$ such that $Kx = B_\ast x$ and therefore

$$\|x - tB_\ast x - (1 - t)B_0x\| < c\|x\|$$  

(3.12)

But $B_1 \leq B \leq B_2$ for $B = tB_\ast + (1 - t)B_0$ and consequently (3.10) holds. This contradicts (3.12) and our claim is valid. Hence, the conclusions of (a), (b) and (c) follow from Theorems 1 and 3.1 [16], respectively. □
Remark 3.3. Theorem 3.4 is applicable if $B_0$ is compact and $M+N$ is the sum of a $k$-ball-contraction and a monotone map, $k < 1$, or $N$ is compact and $(Mx - My, x - y) \geq -||x - y||^2$, etc. When $B_0$ and $N$ are compact, $M = 0$ and $|N - K| = 0$, the solvability of $x - Nx = f$ in part (a) has been proven by Krasnoel’skii-Zabreiko [8] and in a less general form by Perov [20], using completely different arguments.

Finally, we shall consider Eq. (1.2) when $D(\Lambda)$ is not a linear subset of $X$ and $\Lambda : D(\Lambda) \subset X \to Y$ is such that

$$ (A+C)^{-1} : Y \to D(\Lambda) \subset X \text{ is surjective and } \|(A+C)^{-1}y\| \leq K(||y|| + 1) \quad (3.13) $$

for some bounded map $C : X \to Y$, each $y \in Y$ and some constant $K > 0$. Condition (3.13) is satisfied if, e.g., $Y = X$ and $C = \lambda I$, $\lambda > 0$, and $A$ is $m$-accretive (cf. [1]). In applications considered in part II (3.13) holds with $Y \neq X$.

**Theorem 3.5.** [17]. Let (3.13) hold and $N : D(\Lambda) \subset X \to Y$ be such that for some constants $a > 0$, $b > 0$ with $\delta K a < 1$, $\delta = \max ||P_n||$, (3.14)

$$ ||Nx - Cx|| \leq a||x|| + b \quad \text{for} \quad x \in D(\Lambda). \quad (3.14) $$

Suppose that $T = I + (N - C)(A + C)^{-1} - \mu C(A + C)^{-1}$ is $A$-proper w.r.t. $\Gamma_0 = \{X_n, P_n\}$ for $Y$ and $\mu \in [0, 1)$ and $T_0$ satisfies condition (*). Then $(A + N)(D(\Lambda)) = Y$.

**Proof.** It is easy to see that Eq. (1.2) is solvable if and only if so is the equation $T_0y = f$ in $Y$. In view of Corollary 2.1, with $A = I$ and $G = -C(A + C)^{-1}$, it suffices to show that $||(N - C)(A + C)^{-1}|| < 1/\delta$. But, this follows easily from (3.13)–(3.14) since

$$ \limsup_{||y|| \to \infty} \frac{||(N - C)(A + C)^{-1}y||}{||y||} \leq \limsup_{||y|| \to \infty} \frac{b + a||(A+C)^{-1}y||}{||y||} \leq aK < 1/\delta. \quad \square $$

Next, we shall give an extension of Theorem 3.5 when (3.13) does not hold. We need

**Definition 3.1.** A homotopy $H : [0, 1] \times D \to Y$, $D \subset X$, is said to satisfy condition (+) if $\{x_n\}$ is bounded in $X$ whenever $H(t_n, x_n) \to f$, $t_n \in [0, 1]$.

**Theorem 3.6.** [17]. Let $A, N : D(\Lambda) \subset X \to Y$ and $C : X \to Y$ be nonlinear maps, $C$ and $N$ be bounded and $(A + C)^{-1} : Y \to D(\Lambda)$ be bounded and surjective. Suppose that $H(t, x) = Ax + tNx + (1 - t)Cz$, $t \in [0, 1]$ satisfies condition (+), $F_t = I + t(N - C)(A + C)^{-1}$ is $A$-proper w.r.t. $\Gamma_0 = \{Y_n, P_n\}$ for each $t \in [0, 1)$ and $F_t$ satisfies condition (*). Then $(A + N)(D(\Lambda)) = Y$.

**Proof.** Let $f \in Y$ be fixed. Condition (+) implies that the set $U = \{x \in D(\Lambda) | H(t, x) = tf \text{ for some } t \in [0, 1]\} \subset B(0, R_1)$ for some $R_1 > 0$. Then $x = (A+C)^{-1}y \in U$ whenever $F(t, y) = tf$ and, since $C$ and $N$ are bounded, there is an $R > 0$ such that

$$ ||y|| \leq ||(N+C)(A+C)^{-1}y|| \leq R.$$
Hence, $F(t,y) \neq tf$ for $(t,y) \in [0,1] \times \partial B(0,R)$. Next, let $\varepsilon_k \in (0,1)$ and $\varepsilon_k \to 1$. By the $A$-properness of $F_t$ for $t \in [0,\varepsilon_k]$, there is an $n_k = n(\varepsilon_k) \geq 1$ such that

$$P_n F(t,y) \neq tP_n f \quad \text{for} \quad t \in [0,\varepsilon_k], \ y \in Y_n \cap \partial B(0,R), n \geq n_k$$

and $n_{k_1} \geq n_k$ if $k_1 \geq k_2$. Hence, for each $k$ fixed and each $n \geq n_k$

$$\text{deg}(P_n H(\varepsilon_k), B(0,R) \cap Y_n, P_n f) = \text{deg}(I, B(0,R) \cap Y_n, 0) \neq 0$$

and therefore $P_n F(\varepsilon_k, y_n) = \varepsilon_k P_n f$ for some $y_n \in B(0,R) \cap Y_n$ and each $n \geq n_k$. Since $F_{\varepsilon_k}$ is $A$-proper, there is an $y_k \in \overline{B}(0,R)$ such that $F(\varepsilon_k, y_k) = \varepsilon_k f$. Then $y_k + (N - C)(A + C)^{-1} y_k = \varepsilon_k f + (1 - \varepsilon_k)(N - C)(A + C)^{-1} y_k \to f$ as $k \to \infty$. Thus by condition (*) for $F_1$, there is an $y \in Y$ such that $F(1, y) = f$ and so $x = (A + C)^{-1} y$ is a solution of $Ax + N x = f$. □

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(Received 06. 04. 1986)