SOME LIMIT THEOREMS FOR ONE TYPE
OF STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We consider limit theorems for linear stochastic integro-differential equations of a special type and we give sufficient conditions for the almost sure convergence of the sequence of their solutions.

1. Introduction

Let us introduce some assumptions. Let \((\Omega, F, P)\) be a complete probability space on which an \(R^n\)-valued standard Wiener process \(W = [(W_t, F_t), t \geq 0]\) is given, where \((F_t, t \geq 0)\) is a filtration satisfying the usual conditions. The non-random functions

\[
\begin{align*}
    a_n &: [0,T] \times R^d \to R^d, \quad b_n : J \times R^d \to R^d, \quad c_n : J \times R^d \to R^d \times R^m, \\
    A_n &\colon [0,T] \times R^d \to R^d \times R^m, \quad B_n : J \times R^d \to R^d \times R^m, \\
    C_n &\colon J \times R^d \to R^d \times R^m \times R^m,
\end{align*}
\]

where \(T = \text{const} > 0, J = \{(t,s) : (t,s) \in [0,T] \times [0,T], s \leq t\}, n = 0,1,\ldots,\) are Borel-measurable in respect to the corresponding \(\sigma\)-fields on their domains. They satisfy the uniform Lipschitz conditions and the restriction on linear growth i.e. there exists a constant \(K > 0,\) such that for all \((t,s) \in J, x \in R^d, y \in R^d,\)

\[
\begin{align*}
    (1) \quad |a_n(t,x) - a_n(t,y)| &\leq K|x - y|, \quad |C_n(t,s,x) - C_n(t,s,y)| \leq K|x - y|, \\
    (2) \quad |a_n(t,x)|^2 &\leq K^2(1 + |x|^2), \quad |C_n(t,s,x)|^2 \leq K^2(1 + |x|^2),
\end{align*}
\]

and analogously for the other functions.

We suppose that \(R^d\)-valued random processes \(\varphi_n = (\varphi_n(t), t \in [0,T]), n = 0,1,\ldots,\) are nonanticipating for every \(t \in [0,T].\) Also, in what follows we suppose that all integrals, ordinary and stochastic, exist as nonanticipating stochastic processes.

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Berger [1] considered a sequence of stochastic integro-differential equations (shortly SIDE) of the type

\begin{align}
X_n(t) &= \varphi_n(t) + \\
&+ \int_0^t [a_n(s, X_n(s)) + \int_0^s b_n(s, u, X_n(u))du + \int_0^s c_n(s, u, X_n(u))dW(u)]ds + \\
&+ \int_0^t [A_n(s, X_n(s)) + \int_0^s B_n(s, u, X_n(u))du + \int_0^s C_n(s, u, X_n(u))dW(u)]ds,
\end{align}

and, under the conditions (1) and (2), he proved the existence and uniqueness of the \(R^2\)-valued, nonanticipating solutions \(X_n = (X_n(t), t \in [0, T]), n = 0, 1, \ldots\). Also, he gave the conditions for the convergence in mean square of the sequence of solutions \(\{X_n\}, n = 1, 2, \ldots\), to the solution \(X_0\) as \(n \to \infty\).

The main problem of this paper is to give sufficient conditions under which the sequence of solutions \(\{X_n\}, n = 1, 2, \ldots\), converges almost surely to the solution \(X_0\) as \(n \to \infty\). We must require a closeness in some sense of the processes \(\varphi_n\), \(n = 1, 2, \ldots\), and the functions \(a_n, b_n, c_n, A_0, B_0, C_0, n = 1, 2, \ldots\), to the process \(\varphi_0\) and the functions \(a_0, b_0, c_0, A_0, B_0, C_0\), respectively. The paper [5] contains some ideas and results about this closeness.

Let us introduce the following conditions:

\begin{align}
(4) \quad &\sup_n \sup_t E[|\varphi_n(t)|^2] < \infty; \\
(5) \quad &\sum_{n=1}^{\infty} E[\sup_t |\varphi_n(t) - \varphi_0(t)|^2] < \infty; \\
(6) \quad &\sum_{n=1}^{\infty} \sup_{(t, s) \in \Pi} \left\{ |a_n(t, x) - a_0(t, x)| + |b_n(t, s, x) - b_0(t, s, x)| + \\
&+ |c_n(t, x) - c_0(t, x)| + |A_n(t, x) - A_0(t, x)| + \\
&+ |B_n(t, s, x) - B_0(t, s, x)| + |C_n(t, s, x) - C_0(t, s, x)| \right\} < \infty,
\end{align}

where \(\Pi = \{(t, s, x) : (t, s) \in J, x \in R\}\).

2. Main results

**Theorem 1.** Let the functions \(a_n, b_n, c_n, A_n, B_n, C_n, n = 0, 1, \ldots\), and the processes \(\varphi_n, n = 0, 1, \ldots\), be defined as above and all preceding conditions be satisfied. Then the sequence of random processes \(\{X_n\}, n = 1, 2, \ldots\), converges almost surely, uniformly in \(t, t \in [0, T]\), to the random process \(X_0\) as \(n \to \infty\).

The conditions of Theorem 1 could be weakened. Following the tradition of the classical theory of stochastic differential equations, we suppose that for each number \(M > 0\) there exists a constant \(L_M > 0\), such that the condition (1) is valid with this constant for all \((t, s) \in J, |x| \leq M, |y| \leq M\). Also, the expectation in (4) does not have to be bounded, and (2) and (5) are satisfied.
Some limit theorems for one type of stochastic integro-differential equations

**Theorem 2.** Let the functions \(a_n, b_n, c_n, A_n, B_n, C_n, n = 0, 1, \ldots,\) and the processes \(\varphi_n, n = 0, 1, \ldots,\) satisfy the preceding conditions and let \((a)\) be valid for \(\Xi' = \{(t, s, x) : (t, s) \in J, |x| \leq M\}\) instead of \(\Pi.\) Then the sequence of random processes \(\{X_n\}, n = 1, 2, \ldots,\) converges almost surely, uniformly in \(t, t \in [0, T],\) to the random process \(X_0\) as \(n \to \infty.\)

### 3. Proofs of theorems

**Proof of Theorem 1.** Denote

\[
\varepsilon_n = E\left\{ \sup_{t \in J} \|b_n(s, X_n(s)) - a_0(s, X_n(s))\|^2 + \right. \\
\left. + |b_n(t, s, X_n(s)) - b_0(t, s, X_n(s))|^2 + |c_n(t, s, X_n(s)) - c_0(t, s, X_n(s))|^2 + \\
+ |A_n(s, X_n(s)) - A_0(s, X_n(s))|^2 + |B_n(t, s, X_n(s)) - B_0(t, s, X_n(s))|^2 + \\
+ |C_n(t, s, X_n(s)) - C_0(t, s, X_n(s))|^2 \right\}, \quad n = 1, 2, \ldots
\]

Since we must find an upper bound for \(E\{ |X_n(t) - X_0(t)|^2 \},\) we will estimate only one integral, and analogously the others. If we apply one of the basic properties of the Itô integrals (see [2], [3]), add some terms, use the Cauchy-Schwarz inequality, the condition (1) and the notation (7), we obtain

\[
E\left\{ \int_0^t \int_0^s [B_n(s, u, X_n(u)) - B_0(s, u, X_0(u))]duW(s) \right\}^2 = \\
= \int_0^t \int_0^s E\left\{ \left| \int_0^s [B_n(s, u, X_n(u)) - B_0(s, u, X_0(u))]du \right|^2 \right\}ds \leq \\
\leq 2 \int_0^t \int_0^s E\{|B_n(s, u, X_n(u)) - B_0(s, u, X_0(u))|^2 + \\
+ |B_0(s, u, X_n(u) - B_0(s, u, X_0(u)))|^2 \}duds \leq \\
\leq t^2E\varepsilon_T + K^2 + \int_0^t E\{|X_n(s) - X_0(s)|^2 \}ds.
\]

Subtracting the equations (3) for \(n = 0\) from (3) and using (8), it is easy to obtain the estimation

\[
E\{ |X_n(t) - X_0(t)|^2 \} \leq 7\delta_n + \alpha \left[ \varepsilon_nT + K^2 \int_0^t E\{|X_n(s) - X_0(S)|^2 \}ds \right],
\]

where \(\alpha = 14[T^3/2 + 3T^2/2 + 2T + 1].\) By the Gronwall-Bellman inequality we have

\[
E\{ |X_n(t) - X_0(t)|^2 \} \leq (7\delta_n + \alpha \varepsilon_nT)e^{\alpha K^2t}.
\]
We will apply the well-known inequality for stochastic Itô integrals:

$$E\left\{ \sup_{t \in [0,T]} \int_0^t X(s, \omega) dW(s) \right\}^2 \leq 4 \int_0^T E\{|X(t, \omega)|^2\} dt,$$

if the last integral is finite (see [2], [3]).

For example, from (9) it follows that

$$E\left\{ \sup_{t \in [0,T]} \int_0^t \left[ B_n(s, u, X_n(u)) - B_0(s, u, X_0(u)) \right] dW(s) \right\}^2 \leq$$

$$\leq 4 \int_0^T E\left\{ \left| \int_0^s \left[ B_n(s, u, X_n(u)) - B_0(s, u, X_0(u)) \right] du \right|^2 \right\} ds \leq$$

$$\leq 4T^2 \left[ \varepsilon T + K^2 \int_0^T E\{|X_n(s) - X_0(s)|^2\} ds \right].$$

We can find upper bounds for the other integrals similarly. So we have

$$E\left\{ \sup_{t \in [0,T]} |X_n(t) - X_0(t)|^2 \right\} \leq 7\delta_n + \beta \left[ \varepsilon_n T + K^2 \int_0^T E\{|X_n(t) - X_0(t)|^2\} dt \right],$$

where $\beta = 14[T^3/2 + 3T^2 + 6T + 4]$. Thus, from (10) and the last relation, it follows that

$$E\left\{ \sup_{t \in [0,T]} |X_n(t) - X_0(t)|^2 \right\} \leq c_1\delta_n + c_2\varepsilon_n,$$

where $c_1$ and $c_2$ are corresponding constants. By Chebyshev’s inequality, for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} P\left\{ \sup_{t \in [0,T]} |X_n(t) - X_0(t)| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} E\left\{ \sup_{t \in [0,T]} |X_n(t) - X_0(t)|^2 \right\} \leq$$

$$\leq \frac{c_1}{\varepsilon^2} \sum_{n=1}^{\infty} \delta_n + \frac{c_2}{\varepsilon^2} \sum_{n=1}^{\infty} \varepsilon_n < \infty,$$

because, from the conditions (5) and (6), the series $\sum_{n=1}^{\infty} \delta_n$ and $\sum_{n=1}^{\infty} \varepsilon_n$ are convergent. By the Borel-Cantelli’s lemma and Weierstrass’ uniform convergence theorem, it follows that the sequence of random processes $\{X_n\}, n = 1, 2, \ldots$, converges almost surely, uniformly in $t, t \in [0, T]$, to the random process $X_0$ as $n \to \infty$. Thus Theorem 1 is proved.

**Proof of Theorem 2.** For a chosen number $M > 0$, let us denote

$$\psi_M(x) = \begin{cases} x, & \text{if } |x| < M \\ M \text{sgn } x, & \text{if } |x| \geq M \end{cases}$$

$$\varphi^M_n(t) = \psi_M(\varphi_n(t)), a^M_n(t, x) = a^M_n(t, \psi_M(x)), b^M_n(t, s, x) = b^M_n(t, s, \psi_M(x)),$$
and analogously for $c_n, A_n, B_n, C_n, n = 0, 1, \ldots$. Let $X^M_n(t)$ be a solution of the SIDE

$$X^M_n(t) = \varphi^M_n(t) + 
\int_0^t \left[ a^M_n(s, X^M_n(s)) + \int_0^s b^M_n(s, u, X^M_n(u)) du + \int_0^s c^M_n(s, u, X^M_n(u)) dW(u) \right] ds +
\int_0^t \left[ A^M_n(s, X^M_n(s)) + \int_0^s B^M_n(s, u, X^M_n(u)) du +
\int_0^s C^M_n(s, u, X^M_n(u)) dW(u) dW(s) \right], \quad n = 0, 1, \ldots
$$

These solutions exist by the existence and uniqueness theorem. Also, all conditions of Theorem 1 are satisfied and thus the sequence $\{X^M_n\} n = 1, 2, \ldots$, converges almost surely to the process $X_0$ as $n \to \infty$. From that fact, we will show that $\{X_n\}, n = 1, 2, \ldots$, converges almost surely to $X_0$ as $n \to \infty$. Let

$$\tau^M_n = \begin{cases} \inf \{ t : |X^M_n(t)| > M \} \\
T, \text{ if } |X_n(t)| \geq M \text{ for all } t 
\end{cases}
$$

be stopping times with respect to $(F_t, t \geq 0)$. For each $t, t \in [0, T]$, we can find a sufficiently large $M$, such that $\tau^M_n > t, n = 0, 1, \ldots$, almost surely. Since there exists a stopping time $\tau^z = \epsilon_n$, (see [4]) then on the interval $[0, \tau^z]$ we have

$$\varphi^M_n(t) = \varphi_n(t), \quad a^M_n(t, X^M_n(t)) = a_n(t, X_n(t)), \quad b^M_n(t, s, X^M_n(s)) = b_n(t, s, X_n(s)),
$$

and similarly for $c_n, A_n, B_n, C_n, n = 0, 1, \ldots$. Thus on the interval $[0, \tau^z]$ the sequence $\{X_n\} n = 1, 2, \ldots$, converges almost surely to $X_0$ as $n \to \infty$. Since

$$\lim_{M \to \infty} P\{\tau^M = T\} = 1$$

(see [1]), it follows that the sequence $\{X_n\}, n = 1, 2, \ldots$, converges almost surely, uniformly in $t, t \in [0, T]$, to $X_0$ as $n \to \infty$. Thus the proof is complete.

Remark. Theorem 1 and Theorem 2 can be proved if the coefficients of the SIDE (3) are random functions. In this case, $a_n$ and $A_n$ must be nonanticipating in $s$ for each $x$, and $b_n, B_n, c_n, C_n$ must be nonanticipating in $s$ for all $(t, x)$. Also, the conditions (1), (2) and (6) must be satisfied almost surely.

REFERENCES


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